INFINITE SETS OF SOLUTIONS AND ALMOST SOLUTIONS OF THE EQUATION N·M = REVERSAL(N·M) II

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Abstract:

Motivated by their intrinsic interest and by applications to the study of numeric palindromes, we discover a method for producing infinite sets of solutions and almost solutions of the equation, $N \times M = reversal(N \times M)$, our results are valid in a general numeration base $b > 2$.

Introduction:

When doing multiplication, let alone any calculation, one generally desires to have their results as quickly as possible. To accomplish this, we use numerical methods as well as predictions from experience and memory to save time. Realistically, most 5th graders could calculate 8×9999999999 using traditional methods, pen, paper along with patience. Much alike of the 5th grader, computers often also employ brute force methods to make even the simplest calculations. However, enough digits could simply render computers to run out of memory and force the 5th grader to make an error. After all, regardless of all the advances in software and technology, computers, in essence, remain to be very simple binary devices that prefer things in 0s and 1s-- off and on respectively. Speaking of computer's requirement of base-2, on the other "hand," with a natural tendency to have 10 fingers, humans tend to favor base-10 number system-- meaning our 5th grader would struggle computing a similar calculation of 5×6666666666 in base-7. Needless to say, if provided with enough digits, both the computer and the 5th grader would cave in defeat.

In this research, motivated by challenges previously mentioned, along with their intrinsic interest and by the applications of numeric palindromes (numbers that remain same when its digits are reversed), and other sequences of integers, we discover a method for producing infinite sets of solutions and almost solutions to the equation

$$N \times M = reversal(N \times M).$$

Equation 1

Where, if N is an integer written in any base b, and so $reversal(N)$ simply is N's digits written in reverse, e.g. $N = 81$ means $reversal(N) = 18$. At the same time base b is simply the number or combination of digits in a number system that could be used to count, e.g. base-2, in other words binary system $b = 2$, only has 2 digits: $0,1$, while base-5, or $b = 5$ contains the digits $0,1,2,3,4$ for a total of 5 digits to count.

An almost solution of the equation 1 is a pair of integers (N, M) for which above equality holds up to a few digits for which we understand their position. Our results are valid in a general numeration base $b > 2$ and complement the results in our preceding research on this subject [4]. Recently, one of us showed in *Infinite sets of b-additive and b-multiplicative Ramanujan-Hardy numbers* [2] that, in any numeration base-b, for any integer N not divisible by b, e.g. 21 in base-5 is not divisible by 5, the equation 1 has an infinite set of solutions (N, M). Nevertheless, as one can see from De Geest's *World of numbers* [3], finding explicit values for M can be difficult from a computational point of view, even for small values of N, e.g. $N = 81$ as expressed previously from the 5th graders perspective when we use different bases. We present in many numeration bases explicit infinite families of solutions of equation 1 that complement and are also independent of our previous research on the subject [4].

Another application of our results, aside making the life of 5th graders tasked with ghastly multiplications easier, may appear in the study of the classes of *b*-multiplicative and *b*-additive Ramanujan-Hardy numbers, recently introduced by Nitica in *About some relatives of the taxicab number* [1]. The first class consists of all integers N for which there exists an integer M such that $S_b(N)$, in other words the sum of base-b digits of N, times M, multiplied by the reversal of the product, is equal to N. The second class consists of all integers N for which there exists an integer M such that $S_b(N)$, times M, added to the reversal of the product, is again equal to N. In more mathematical notation, as shown by Nitica [1,2], the solutions of equation 1 for which we can compute the sum of digits of $S_b(N) \times M + reversal(S_b(N) \times M)$, or of $S_b(N) \times M \times reversal(S_b(N) \times M)$, can be used to find infinite sets of above numbers.

Statements of the Main Results:

The heuristics behind our results is that the product of a palindrome by a small integer still preserves some of the symmetric structure of the palindrome if, in addition, the palindrome has as many digits of 9, the results observed in base-10 can be carried over to an arbitrary numeration base b replacing 9 by the resulting integer of $(b-1)$.

Let us begin by introducing some notation with the initial condition that $b \geq 2$ be a numeration base. If x is a string of digits, let $x^{\wedge k}$ denote the base b integer obtained by repeating x k-times. Finally let $[x]_b$ denote the value of the string x in base b. For example, suppose we have $[2^{\wedge 3}]_5$, meaning our $x = 2$, $k = 3$, and $b = 5$, therefore $[2^{\wedge 3}]_5 = [222]_5$, simply put repeating 2 for a total of 3 times.

Next theorem is one of our main results.

Theorem 1. Let $b \geq 2$ be a numeration base and $0 < A, B, c, d \leq b$ such that $A \times B = [cd]_b$ where $c + d = A$. Then,

$$A^{\wedge k} \times B = \left[cA^{\wedge(k-1)}d\right]_b.$$

Theorem 1

Let us illustrate this with an example that is more useful to our 5th grader. Suppose $b = 10$, in other words our traditional base-10 system, and $(A, B) = (9,9)$, meaning $9 \times 9 = [cd]_{10} = [81]_{10}$ such that $c + d = 8 + 1 = 9$. Now, for any k we have that $9^{\wedge k} \times 9 = \left[c9^{\wedge(k-1)}d\right]_{10} = \left[89^{\wedge(k-1)}1\right]_{10}$. If our $k = 3$, then we have that $9^{\wedge 3} \times 9 = \left[89^{\wedge(3-1)}1\right]_{10}$, which results in $999 \times 9 = [8991]_{10}$. This should allow our 5th grader to seriously impress their instructor or improve computation times or at the very least make for a great party trick. Table 1 showcases the results for several small values of k to showcase the relative ease in which one can produce results for this example with varying k.

Corollary 1. Let $b \geq 2$ be a numeration base. Let $0 < A, B, c, d, \alpha \leq b$ such that $A \cdot B = [cd]_b$ and $c + d = \alpha$. Then,

$$A^{\wedge k}B = \left[c\alpha^{\wedge(k-1)}d\right]_b = AB^{\wedge k}.$$

Corollary 1

Theorem 2. Let $b \geq 2$ be a numeration base. Then, the following pairs satisfy the hypothesis of Theorem 1.

$$(AB) = \left[\left((b-1)(b-k)\right)\right]_b, 1 \leq k \leq b.$$

Theorem 2

Corollary 2. Let $b \geq 2$ be a numeration base. Then $[(b-1)(b-2)]_b$ consequently satisfy the hypothesis of Theorem 1,

$$(b-1)^{\wedge k}(b-2) = \left[(b-3)(b-1)^{\wedge(k-1)}2\right]_b.$$

Corollary 2

Let us illustrate once more to our 5th grader, now with combination of the results of Theorems 1 and 2, in a different base. Suppose we have that $b = 7$, then following Theorem 2, $b - 1 = 6$ and $b - 2 = 5$, thus $A = 6$, $B = 5$ and $[cd]_b = [42]_7$. To clarify, we note that $[6 \cdot 5]_7 = [42]_7$ (or in other terms, $6 \cdot 5 = [30]_{10} = [42]_7$) and $[4 + 2]_7 = 6$, hence $[c + d]_b = A$. So, if our $k = 14$, following our recipe we have that $[66666666666666 \cdot 5]_7 = \left[cA^{\wedge(k-1)}d\right]_b = [466666666666662]_7$.

Conclusions:

If our results are to be confirmed using hand techniques, especially our last example, one would note the significant amount of time it requires to compute by hand, especially if we are operating in any base other than base-10. Therefore, not only our method produces infinite sets of solutions and almost solutions of the equation $N \times M = reversal(N \times M)$ in a general numeration base $b > 2$, moreover it has the potential to save a significant amount of computation time for 5th graders, as well as computers while doing multiplication with significantly many integer digits.