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Abstract. We study a natural generalization of covering projections defined in terms of unique lifting properties. A map $p : E \to X$ has the *continuous path-covering property* if all paths in X lift uniquely and continuously (rel. basepoint) with respect to the compact-open topology. We show that maps with this property are closely related to fibrations with totally path-disconnected fibers and to the natural quotient topology on the homotopy groups. In particular, the class of maps with the continuous path-covering property lies properly between Hurewicz fibrations and Serre fibrations with totally path-disconnected fibers. We extend the usual classification of covering projections to a classification of maps with the continuous path-covering property in terms of topological π_1: for any path-connected Hausdorff space X, maps $E \to X$ with the continuous path-covering property are classified up to weak equivalence by subgroups $H \leq \pi_1(X, x_0)$ with totally path-disconnected coset space $\pi_1(X, x_0)/H$. Here, *weak equivalence* refers to an equivalence relation generated by formally inverting bijective weak homotopy equivalences.

1. Introduction. The breadth of the applications of covering space theory has motivated the development of many useful generalizations. Perhaps most historically prominent is Spanier's classical treatment of Hurewicz fibrations with totally path-disconnected fibers [35, Chapter 2]. Unfortunately, since fibrations are defined abstractly in terms of homotopy lifting with respect to arbitrary spaces, it is not possible to extend the usual classification of covering projections to classify these maps up to homeomorphism, i.e. deck transformation. Indeed, there exist bijective fibrations $E \to X$ of metric spaces which are not homeomorphisms [35, Example 2.4.8].

In this paper, we identify and classify a natural class of maps that lies properly between Hurewicz and Serre fibrations with totally path-disconnected fibers, namely, those maps that satisfy the following property: a map

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$p : E \to X$ has the *continuous path-covering property* if for every $e \in E$, all paths $\alpha : ([0,1],0) \to (X, p(e))$ have a unique lift $\widetilde{\alpha} : ([0,1],0) \to (E, e)$ such that the lifting function $\alpha \mapsto \widetilde{\alpha}$ is continuous with respect to the compact-open topology on based path spaces (Definition 3.1).

We note that there is no single, best, or end-all generalized covering space theory since one must choose the properties and structures that one cares about according to intended applications. The literature on the subject is vast, and we do not attempt to give a complete survey here. We briefly mention that there are many natural approaches closely related to shape theory such as R. H. Fox's theory of overlays [26]. Similar approaches using pro-groups or other algebraic structures in place of the usual fundamental group are considered and compared in [1, 29, 30, 37] and the references therein. Covering theories for categories other than the usual topological category, such as uniform spaces [3, 10] and topological groups [2], have also appeared. The approach we take is motivated by the ongoing development of topological methods for studying and applying the algebraic and topological properties of fundamental groups. In particular, our approach is most closely related to the theory of semicoverings [5, 25, 34]. Semicoverings and topologized fundamental groups have been used to fill in longstanding gaps in general topological group theory [6], which has not been achieved using purely topological methods. We expect similar applications to follow from the strengthened relationship between covering-theoretic methods and topological group theory developed in the current paper.

Hurewicz fibrations with totally path-disconnected fibers form a class of maps with many nice internal properties that includes all inverse limits of covering projections. In recent work, the authors of [14] have identified sufficient compactness conditions on the fibers of such a fibration $p : E \to X$ to ensure that p is equivalent to an inverse limit of finite-sheeted covering maps. When local triviality or compactness conditions on the fibers are not assumed, it becomes an onerous task to verify that a given map $p : E \to X$ with unique lifting of all paths (i.e. with the *path-covering property*) is a Hurewicz fibration. In fact, when X is the closed unit disk and E is locally path-connected, this verification is equivalent to a curiously difficult open problem posed by Jerzy Dydak [16] (see Problem 4.6 below). The apparent difficulty stems from the fact that if one does not already know that p is an inverse limit of fibrations, then one must verify the homotopy lifting property with respect to *all* topological spaces Z. Even if one restricts to a convenient category of spaces, one must still address spaces Z in which convergent nets are not always realized as the endpoints of some convergent net of paths. Since our goal is to obtain a theory that supports application of (topologized) homotopy groups, it is clear that we must weaken the highly demanding definition of a Hurewicz fibration.

Serre fibrations with totally path-disconnected fibers form a significantly larger class of maps than their Hurewicz counterparts. Included among these are the generalized regular covering maps defined originally by Fischer and Zastrow in [24] and later in [11] using a different but ultimately equivalent approach. Such maps were extended to the non-regular case in [7] and characterized completely within the π_1-subgroup lattice for metric spaces in [9]. Such generalized covering maps provide a theory which has been proven to retain the largest possible π_1-subgroup lattice among any other theories that employ homotopy lifting [7]. For instance, such maps can retain non-trivial information about the π_1-subgroup lattice even for spaces with trivial shape-type. Consequently, the intended application of this locally path-connected approach is to provide a highly refined theory to aid the progressive work on the infinitary-algebraic structure of homotopy groups of Peano continua and other locally path-connected spaces.

In Section 3, we develop the basic theory of the continuous path-covering property. We observe that, just as with fibrations, maps with the continuous path-covering property are closed under composition, infinite products, pullbacks, and inverse limits. Moreover, these maps even enjoy the two-out-of-three property (see Lemma 3.5). In Section 4, we give the following comparison of maps with unique lifting properties.

THEOREM 1.1. *Consider the following properties of a map $p : E \to X$:*

(1) *p is a Hurewicz fibration with the unique path-lifting property,*
(2) *p has the continuous path-covering property,*
(3) *p is a Serre fibration with the unique path-lifting property,*
(4) *p has the path-covering property.*

Then (1)⇒(2)⇒(3)⇒(4).

Following the proof of Theorem 1.1, we provide examples to show that the conditionals (1)⇒(2) and (2)⇒(3) are not reversible. Hence, the continuous path-covering property lies properly between Hurewicz and Serre fibrations with totally path-disconnected fibers. To promote the fundamental nature of Dydak's Problem, referenced above, we show that an affirmative answer to this problem is equivalent to the converse of (3)⇒(4). We also identify a consequence of a positive answer to Dydak's Problem within the context of classical covering space theory (Corollary 4.10).

In Section 5, we consider the homotopy groups $\pi_n(X, x_0)$ with the natural quotient topology inherited from the compact-open topology on the nth iterated loop space $\Omega^n(X, x_0)$ [21, 27]. With this topology, the homotopy groups become functors to the category of quasitopological groups and continuous homomorphisms. We show that a map $p : E \to X$ with the continuous path-covering property induces a closed embedding on fundamental groups and an isomorphism of quasitopological abelian groups on the higher homotopy groups (Theorem 5.2). Moreover, for any $e \in E$, the coset space $\pi_1(X, x_0)/p_{\#}(\pi_1(E, e))$ is totally path-disconnected and is homeomorphic to the fibers of p when the path endpoint evaluation map $\mathrm{ev}_1 : P(E, e_0) \to E$, $\mathrm{ev}_1(\alpha) = \alpha(1)$ is a topological quotient map (Theorem 5.4). The topological property "$\mathrm{ev}_1 : P(Z, z) \to Z$ is quotient" of a space Z is a generalization of a space being "path-connected and locally path-connected", which arises naturally within our work.

Section 6 is dedicated to a proof of our classification theorem. Just as with Serre/Hurewicz fibrations with unique path lifting, it is not possible to extend the traditional classification of covering projections in a way that classifies all maps with the continuous path-covering property up to *homeomorphism* (see Example 6.5). Therefore, to identify a suitable and practical classification, we employ a technique from model category theory [33], namely, "localization at the weak homotopy equivalences", which refers to the formal inversion of weak homotopy equivalences to generate an equivalence relation. A *simple weak equivalence* between two maps $p_i : E_i \to X$, $i \in \{1, 2\}$, with the continuous path-covering property consists of a commutative diagram

$$\begin{array}{ccccc} E_1 & \xleftarrow{f_1} & E_3 & \xrightarrow{f_2} & E_2 \\ & {\scriptstyle p_1}\searrow & \downarrow{\scriptstyle p_3} & \swarrow{\scriptstyle p_2} & \\ & & X & & \end{array}$$

where p_3 also has the continuous path-covering property and f_1, f_2 are *bijective* weak homotopy equivalences (in fact, they necessarily induce topological isomorphisms on all homotopy groups). Then p_1, p_2 are *weakly equivalent* if they are connected by a finite sequence of simple weak equivalences. Our main classification result is the following.

THEOREM 1.2. *Suppose (X, x_0) is a path-connected Hausdorff space and $H \leq \pi_1(X, x_0)$. There exists a map $p : (E, e_0) \to (X, x_0)$ with the continuous path-covering property, unique up to weak equivalence, such that $p_{\#}(\pi_1(E, e_0)) = H$ if and only if $\pi_1(X, x_0)/H$ is totally path-disconnected. Moreover,*

(1) *among the maps $p : E \to X$ for which $\mathrm{ev}_1 : P(E, e) \to E$ is quotient, the maps p are classified up to equivalence,*
(2) *every map $p : E \to X$ with the continuous path-covering property is weakly equivalent to another $p' : E' \to X$ where $\mathrm{ev}_1(E', e') \to E'$ is quotient.*

The proof of the existence portion of Theorem 1.2 requires a careful analysis of the natural quotient construction of covering spaces. A surprising consequence of our proof of (2) of Theorem 1.2 is that two weakly equivalent maps can always be related by a single simple weak equivalence. By employing topologized fundamental group theory [8], another immediate consequence of this theorem is that any space X whose fundamental group naturally injects into the first shape group admits a '*universal weak equivalence class*, i.e. a map $E \to X$ with the continuous path-covering property where E is simply connected. For instance, all one-dimensional metric spaces and planar sets admit such a map.

If $p : E \to X$ has the continuous path-covering property and non-discrete fibers, then E will rarely be locally path-connected. In a sense, this is the price one must pay to ensure that paths lift continuously. However, (2) of Theorem 1.2 still allows for a certain level of topological control, namely, that we may always choose E to have the slightly weaker property that $ev_1 : P(E, e_0) \to E$ is quotient. In summary, (1) and (2) of Theorem 1.2 show that our classification up to weak equivalence restricts to a more traditional classification "up to homeomorphism" when we restrict to the category of spaces E for which $\mathrm{ev}_1 : P(E, e) \to E$ is quotient.

Finally, in Section 7, we prove the following theorem, which identifies natural situations where a weak equivalence class may be represented by an inverse limit of covering projections/semicovering maps.

THEOREM 1.3. *Suppose $p : (E, e_0) \to (X, x_0)$ has the continuous path-covering property where X is locally path-connected and Hausdorff, and suppose $H = p_{\#}(\pi_1(E, e_0))$ is a normal subgroup of $\pi_1(X, x_0)$.*

(1) *If $\pi_1(X, x_0)/H$ is a compact group, then p is weakly equivalent to an inverse limit of finite-sheeted, regular covering projections, restricted to a path component of the domain.*
(2) *If E is simply connected, i.e. $H = 1$, and $\pi_1(X, x_0)$ is locally compact, then p is weakly equivalent to an inverse limit of semicovering maps, restricted to a path component of the domain.*

In Section 8, we conclude the paper with a single diagrammatic summary of our results. This diagram, which incorporates our main results, illustrates the relationships between the maps considered throughout this paper as well as the relationships between these maps and the topological-algebraic properties of $\pi_1(X, x_0)$.

2. Notation and preliminaries

2.1. Mapping spaces and path spaces. If X and Y are spaces, Y^X denotes the set of continuous maps $X \to Y$ with the compact-open topology, and for $A \subseteq X$, $B \subseteq Y$, $(Y, B)^{(X,A)}$ denotes the subspace of relative maps $f \in Y^X$ satisfying $f(A) \subseteq B$. In particular, $\Omega^n(X, x)$, $n \in \mathbb{N}$, will denote the relative mapping space $(X,x)^{(I^n,\partial I^n)}$ where $I=[0,1]$ is the closed unit interval. The constant map $X\to Y$ at a point $y\in Y$ will be denoted c_y. As special cases, we define $P(X)=X^I$ and $P(X,x)=(X,x)^{(I,0)}$. If $\alpha,\beta:I\to X$ are paths such that $\alpha(1)=\beta(0)$, then $\alpha\cdot\beta:I\to X$ denotes the standard concatenation and $\alpha^-(t)=\alpha(1-t)$ denotes the reverse path of α.

If $f:X\to Y$ is a map, then $P(f):P(X)\to P(Y)$ denotes the induced map $P(f)(\alpha)=f\circ\alpha$, which also restricts to a map $P(X,x)\to P(X,f(x))$. The *endpoint evaluation map* is the map $\mathrm{ev}_1:P(X,x)\to X$, $\mathrm{ev}_1(\alpha)=\alpha(1)$, which is continuous and onto when X is path-connected. If X is locally path-connected, then ev_1 is an open surjection. Frequently, we will refer to the following property of X: "$\mathrm{ev}_1:P(X,x)\to X$ is a topological quotient map" (this property holds for all points $x\in X$ if it holds for at least one). The property of ev_1 being a quotient map is a natural generalization of the joint property "path-connected and locally path-connected". Path-connected and non-locally-path-connected spaces for which ev_1 is quotient include all non-locally-path-connected, contractible spaces and many spaces used in the theory and applications of generalized covering space theories and topologized fundamental groups; cf. [5, 6].

We refer to [35] as a standard reference on covering space theory. We consider the following properties, each of which is held by all covering projections.

Definition 2.1. Let E and X be topological spaces.

(1) A map $p:E\to X$ has the *unique path-lifting property* if for each $e\in E$, the induced map $P(p):P(E,e)\to P(X,p(e))$ is injective.
(2) A map $p:E\to X$ has the *path-covering property* if for each $e\in E$, the induced map $P(p):P(E,e)\to P(X,p(e))$ is bijective.

We will always assume the spaces E and X are non-empty and path-connected. If $p:E\to X$ has the path-covering property, $E\neq\emptyset$, and X is path-connected, then p must necessarily be surjective.

2.2. Topologized homotopy groups. The nth homotopy group $\pi_n(X,x_0)$ will be equipped with the natural quotient topology inherited from $\Omega^n(X,x_0)$ so that the natural map $\pi:\Omega^n(X,x_0)\to\pi_n(X,x_0)$, $\pi(\alpha)=[\alpha]$ sending a map to its homotopy class is a topological quotient map. It is known that $\pi_n(X,x_0)$ is a quasitopological group in the sense that inversion is continuous and left and right translations $[\alpha]\mapsto[\alpha][\beta]$ and $[\alpha]\mapsto[\beta][\alpha]$ for fixed $\beta\in\Omega^n(X,x_0)$ are also continuous. Although $\pi_n(X,x)$ can fail to be a topological group for any $n\geq 1$ [20, 21], it is a homogeneous space, which is discrete if X is locally contractible [12, 27].

If $f:(X,x)\to(Y,y)$ is a based map, then the homomorphism $f_\#:\pi_n(X,x)\to\pi_n(Y,y)$ is continuous. An isomorphism in the category of quasitopological groups is a group isomorphism which is also a homeomorphism. If f induces an isomorphism $f_\# : \pi_n(X,x) \to \pi_n(Y,y)$ of quasitopological groups for all $n \geq 1$, then we call f a *weak topological homotopy equivalence.*

If $H \leq \pi_1(X,x_0)$ is a subgroup, the coset space $\pi_1(X,x_0)/H$ inherits the quotient topology from $\pi_1(X,x_0)$. The translation homeomorphism $[\alpha] \mapsto [\alpha\cdot\beta]$ of $\pi_1(X,x_0)$ descends to a homeomorphism $H[\alpha] \mapsto H[\alpha\cdot\beta]$ on $\pi_1(X,x_0)/H$. Hence, $\pi_1(X,x_0)/H$ is a homogeneous space, which is T_1 (resp. discrete) if and only if H is closed (resp. open). We refer to [8] for more on π_1 with the quotient topology.

2.3. Fibrations with the unique path-lifting property

Definition 2.2. A map $p : E \to X$ has the *homotopy lifting property* with respect to a space Z if for every pair of maps $f : Z \to E$, $g : Z \times I \to X$ such that $p \circ f(z) = g(z,0)$, there is a map $\widetilde{g} : Z \times I \to E$ such that $p \circ \widetilde{g} = g$. A *Hurewicz fibration* is a map with the homotopy lifting property with respect to all topological spaces. A *Serre fibration* is a map with the homotopy lifting property with respect to I^n for all $n \geq 0$.

Every covering projection in the classical sense is a Hurewicz fibration with discrete fibers, and every Hurewicz fibration is a Serre fibration.

Lemma 2.3 ([35, proof of 2.2.5]). *A Serre fibration $p : E \to X$ has the unique path-lifting property if and only if every fiber of p is totally path-disconnected.*

If a map $p : E \to X$ has the path-covering property and also the homotopy lifting property with respect to I, then all path-homotopies in X lift uniquely (rel. basepoint) to path-homotopies in E. Hence, standard arguments in covering space theory give the following lemma.

Lemma 2.4. *Suppose $p : E \to X$ has the path-covering property and the homotopy lifting property with respect to I, e.g. if p is a Serre fibration with totally path-disconnected fibers. Then for all $e \in E$,*

(1) *the induced homomorphism $p_\# : \pi_1(E,e) \to \pi_1(X,p(e))$ is injective,*
(2) *the unique lift $\widetilde{\alpha} \in P(E,e)$ of a loop $\alpha \in \Omega(X,p(e))$ is a loop based at e if and only if $[\alpha] \in p_\#(\pi_1(E,e))$.*

Moreover, if $p_\# : \pi_1(E,e) \to \pi_1(X,p(e))$ is surjective, then p is a continuous bijection.

3. Maps with the continuous lifting property.

The goal of this section is to develop the basic properties of maps with the following property.

Definition 3.1. A map $p : E \to X$ has the *continuous path-covering property* if for every $e \in E$, the induced function $P(p) : P(E,e) \to P(X,p(e))$ is a homeomorphism.

Remark 3.2. Certainly, we have:

(1) continuous path-covering $\Rightarrow$ (2) path-covering $\Rightarrow$ (3) unique path-lifting.

However, none of the reverse implications hold in general. For instance, any restriction of a covering projection which is not a covering projection itself satisfies (3) but not (2). The generalized universal covering of the Infinite Earring Space constructed in [24] satisfies (2) but not (1); it is described in more detail below (see Example 4.5).

Proposition 3.3. *If $p : E \to X$ has the unique path-lifting property, then for every $x \in X$ the fiber $p^{-1}(x)$ is totally path-disconnected (and hence T_1). Moreover, if X is T_1, then so is E.*

Proof. If $p^{-1}(x)$ admitted a non-constant path $\alpha : I \to p^{-1}(x)$, then α and the constant path at $\alpha(0)$ are distinct lifts of $p \circ \alpha$, a violation of unique path lifting. Every totally path-disconnected space is T_1 since a non-T_1 space must contain a homeomorphic copy of a non-discrete 2-point space, which is necessarily path-connected. Moreover, if $p(e) = x$ and $\{x\}$ is closed, then $\{e\}$ is closed in the closed fiber $p^{-1}(x)$ and thus closed in E. ∎

Recall that an infinite product of covering projections need not be a covering projection; any infinite power of the exponential map $\mathbb{R} \to S^1$ provides an example.

Lemma 3.4. *The following classes of maps are closed under arbitrary direct products:*

(1) *maps with the unique path-lifting property,*
(2) *maps with the path-covering property,*
(3) *maps with the continuous path-covering property,*
(4) *Hurewicz fibrations with totally path-disconnected fibers,*
(5) *Serre fibrations with totally path-disconnected fibers.*

Proof. The first three cases are clear since the based path-space functors $(X, x) \mapsto P(X, x)$ preserve direct products [19, Proposition 3.4.5]. The last two follow from the fact that (i) if maps $p_j : X_j \to Y_j$ have the homotopy lifting property with respect to a space Z, then so does the product map $\prod_j p_j$, and (ii) products of totally path-disconnected spaces are totally path-disconnected. ∎

Covering projections also fail to be closed under function composition. Maps with the continuous path-covering property, in fact, satisfy the "two-out-of-three" condition in the next lemma. Since the various parts of the statement are straightforward to verify from the definitions, we omit the proof.

LEMMA 3.5. *Suppose $f : X \to Y$ and $g : Y \to Z$ are maps of non-empty, path-connected spaces.*

(1) *If f and g have the continuous path-covering property, then so does $g \circ f$.*
(2) *If g and $g \circ f$ have the continuous path-covering property, then so does f.*
(3) *If g is surjective and f and $g \circ f$ have the continuous path-covering property, then so does g.*

Moreover, the statement holds if we replace "continuous path-covering property" with "path-covering property".

REMARK 3.6. Serre/Hurewicz fibrations are closed under composition, and if g and $g \circ f$ are Serre/Hurewicz fibrations, then so is f. Although the authors do not know of a counterexample, we find it unlikely that these classes of maps are closed under the third combination.

The *cone* over a space X is the quotient space $CX = X \times I / X \times \{0\}$. The point $v_0 \in CX$ which is the image of $X \times \{0\}$ is taken to be the basepoint of CX.

DEFINITION 3.7. Let $(J, \leq)$ be a directed set and $K = J \cup \{\infty\}$ be the space obtained by adding one maximal point. Give K the topology generated by the sets $\{k\}$ and $V_k = \{\infty\} \cup \{j \in J \mid j > k\}$ for $k < \infty$. The *directed arc-fan over* J is the cone over K, i.e. the quotient space $F(J) = K \times I / K \times \{0\}$ with basepoint v_0. We will typically identify $K \times (0, 1]$ with its image in $F(J)$.

REMARK 3.8. Standard exponential laws for spaces imply that the convergent nets $\{\alpha_j\}_{j \in J} \to \alpha$ in $P(X, x)$ are in bijective correspondence with based maps $(F(J), v_0) \to (X, x)$. Hence, a map $p : E \to X$ has the continuous path-covering property if and only if for every $e \in E$ and directed set J, the map $F : (E, e)^{(F(J), v_0)} \to (X, p(e))^{(F(J), v_0)}$, $F(\beta) = p \circ \beta$, is a bijection. If X is a metric space, then the compact-open topology on $P(X, x)$ agrees with the topology of uniform convergence and one need only consider maps $F(\omega) \to X$ on the directed fan $F(\omega)$ indexed by the natural numbers.

In the remainder of this section, we show that maps $E \to X$ with the continuous path-covering property also lift maps $Z \to X$ from many other spaces Z both uniquely and continuously.

LEMMA 3.9. *Let Z be a compact Hausdorff space and $z \in Z$. If $p : E \to X$ has the continuous path-covering property, then for every $e \in E$, the induced map $F : (E, e)^{(CZ, v_0)} \to (X, p(e))^{(CZ, v_0)}$, $F(\beta) = p \circ \beta$, is a homeomorphism.*

Proof. Fix $e \in E$. By assumption, $P(p) : P(E, e) \to P(X, p(e))$ is a homeomorphism so it follows from functoriality that $P(p)^Z : P(E, e)^Z \to P(X, p(e))^Z$ is a homeomorphism as well. We call upon some elementary facts related to exponential laws in the category of topological spaces. Since Z is compact Hausdorff, for any based space (A,a), the mapping space $P(A,a)^Z$ is naturally homeomorphic to the relative mapping space $(A,a)^{(Z\times I, Z\times\{0\})}$, which is, in turn, naturally homeomorphic to the based mapping space $(A,a)^{(CZ,v_0)}$. It follows that $F:(E,e)^{(CZ,v_0)}\to(X,p(e))^{(CZ,v_0)}$ is a homeomorphism.

$$\begin{array}{ccc}
P(E,e)^Z & \xrightarrow{\quad P(p)^Z \quad} & P(X,p(e))^Z \\
\cong\downarrow & & \downarrow\cong \\
(E,e)^{(Z\times I,Z\times\{0\})} & \xrightarrow{p^{(Z\times I,Z\times\{0\})}} & (X,p(e))^{(Z\times I,Z\times\{0\})} \\
\cong\downarrow & & \downarrow\cong \\
(E,e)^{(CZ,v_0)} & \xrightarrow{\quad F \quad} & (X,p(e))^{(CZ,v_0)}
\end{array} \qquad \blacksquare$$

Since $I^{n+1}\cong CI^n$, we obtain the following corollary where $\mathbf{0}\in I^n$ denotes the origin.

COROLLARY 3.10. *If $p:E\to X$ has the continuous path-covering property, then for every $e\in E$ and $n\in\mathbb{N}$, the induced map $p^{(I^n,\mathbf{0})}:(E,e)^{(I^n,\mathbf{0})}\to(X,p(e))^{(I^n,\mathbf{0})}$ is a homeomorphism.*

REMARK 3.11. Note that Corollary 3.10 implies that maps with the continuous path-covering property have the homotopy lifting property with respect to I and thus the conclusions of Lemma 2.4 apply to all such maps.

The following lemma generalizes [35, 2.4.5] and is essentially [5, Lemma 2.5]. We give a direct statement and proof that avoids groupoid terminology.

LEMMA 3.12. *If $p:E\to X$ has the continuous path-covering property, $p(e_0)=x_0$, and (Z,z_0) is a based space such that $\mathrm{ev}_1:P(Z,z_0)\to Z$ is quotient, then a map $f:(Z,z_0)\to(X,x_0)$ has a unique continuous lift $\widetilde{f}:(Z,z_0)\to(E,e_0)$ if and only if $f_\#(\pi_1(Z,z_0))\leq p_\#(\pi_1(E,e_0))$.*

Proof. By Corollary 3.10, p uniquely lifts paths and path-homotopies. Hence, the condition $f_\#(\pi_1(Z,z_0))\leq p_\#(\pi_1(E,e_0))$ is equivalent to the well-definedness of the lift function $\widetilde{f}$ with the following standard definition: for $z\in Z$, let $\gamma\in P(Z,z_0)$ be a path ending at z, $\widetilde{f\circ\gamma}\in P(E,e_0)$ be the unique lift of $f\circ\gamma$, and set $\widetilde{f}(z)=\widetilde{f\circ\gamma}(1)$. Since $\mathrm{ev}_1:P(Z,z_0)\to Z$ is quotient, Z is path-connected. This makes the uniqueness of $\widetilde{f}$ clear once we verify continuity. Let $P(p)^{-1}:P(X,x_0)\to P(E,e_0)$ be the continuous lifting homeomorphism, and consider the following diagram for which the commutativity is equivalent to the definition of $\widetilde{f}$:

$$\begin{array}{ccccc} P(Z,z_0) & \xrightarrow{P(f)} & P(X,x_0) & \xrightarrow[\cong]{P(p)^{-1}} & P(E,e_0) \\ \Big\downarrow{\scriptstyle \mathrm{ev}_1} & & & & \Big\downarrow{\scriptstyle \mathrm{ev}_1} \\ Z & & \xrightarrow{\widetilde{f}} & & E \end{array}$$

Since the top composition $P(Z,z_0) \to E$ is continuous and $\mathrm{ev}_1 : P(Z,z_0) \to Z$ is assumed to be quotient, $\widetilde{f}$ is continuous by the universal property of quotient maps. ∎

COROLLARY 3.13. *If $p_1 : (E_1,e_1) \to (X,x_0)$ and $p_2 : (E_2,e_2) \to (X,x_0)$ are maps with the continuous path-covering property such that*

(1) $\mathrm{ev}_1 : P(E_1,e_1) \to E_1$ *and* $\mathrm{ev}_1 : P(E_2,e_2) \to E_2$ *are quotient,*
(2) $(p_1)_\#(\pi_1(E_1,e_1)) = (p_2)_\#(\pi_1(E_2,e_2))$,

then there exists a unique homeomorphism $h : (E_1,e_1) \to (E_2,e_2)$ such that $p_2 \circ h = p_1$.

THEOREM 3.14. *Suppose $p : E \to X$ has the continuous path-covering property, $e \in E$, and (Z,z) is a path-connected space. Consider the map $F : (E,e)^{(Z,z)} \to (X,p(e))^{(Z,z)}$ given by $F(f) = p \circ f$.*

(1) *If Z is contractible, then F is bijective.*
(2) *If Z is contractible and compact Hausdorff, then F is a homeomorphism.*

Proof. Since Z is contractible, there is a section $s : Z \to P(Z,z)$ to the evaluation map $\mathrm{ev}_1 : P(Z,z) \to Z$. Thus, the latter is a quotient map. The injectivity of F follows from the fact that Z is path-connected and p has the unique path-lifting property. Since Z is simply connected and $\mathrm{ev}_1 : P(Z,z) \to Z$ is quotient, Lemma 3.12 applies to give the surjectivity of F.

Note that Lemma 3.9 proves (2) in the case where Z is a cone. For general contractible Z, there is a retraction $r : CZ \to Z$ such that $r(v_0) = z$. This means that for every space (A,a), the induced map $R : (A,a)^{(Z,z)} \to (A,a)^{(CZ,v_0)}$, $R(g) = g \circ r$, is a section and therefore a topological embedding. Consider the naturality diagram

$$\begin{array}{ccc} (E,e)^{(Z,z)} & \xrightarrow{F} & (X,p(e))^{(Z,z)} \\ \Big\downarrow{\scriptstyle R} & & \Big\downarrow{\scriptstyle R} \\ (E,e)^{(CZ,v_0)} & \xrightarrow{p^{(CZ,v_0)}} & (X,p(e))^{(CZ,v_0)} \end{array}$$

where the vertical maps are embeddings and the bottom map is a homeomorphism (recall Lemma 3.9). It follows that F is a topological embedding. ∎

Theorem 3.15. *If $p : E \to X$ has the continuous path-covering property, then so does the map $P(p) : P(E) \to P(X)$ induced on free path spaces.*

Proof. Let $\alpha \in P(E)$, $e = \alpha(0)$. By Corollary 3.10, the induced map $F_2 : (E,e)^{(I^2,\mathbf{0})} \to (X,p(e))^{(I^2,\mathbf{0})}$ is a homeomorphism. Let $A = \{h \in (E,e)^{(I^2,\mathbf{0})} \mid h(t,0) = \alpha(t)\}$ and similarly $B = \{h \in (X,p(e))^{(I^2,\mathbf{0})} \mid h(t,0) = p \circ \alpha(t)\}$. Note that F_2 maps A into B. Since F_2 is surjective, if $h \in B$, then there is a lift $\widetilde{h} \in (E,e)^{(I^2,\mathbf{0})}$ such that $\widetilde{h}(t,0)$ is a path satisfying $\widetilde{h}(0,0) = e$ and $p \circ \widetilde{h}(t,0) = p \circ \alpha(t)$. Since p has the unique path-lifting property, we have $\widetilde{h}(t,0) = \alpha(t)$ and thus $\widetilde{h} \in A$. It follows that F_2 maps A homeomorphically onto B. Restricting the exponential law naturality diagram on the left gives the commutativity of the diagram on the right:

$$\begin{array}{ccc} P(P(E)) & \xrightarrow{P(P(p))} & P(P(X)) \\ \cong\downarrow & & \downarrow\cong \\ E^{I^2} & \xrightarrow{p^{I^2}} & X^{I^2} \end{array} \qquad \begin{array}{ccc} P(P(E),\alpha) & \xrightarrow{P(P(p))} & P(P(X),p\circ\alpha) \\ \cong\downarrow & & \downarrow\cong \\ A & \xrightarrow[(F_2)|_A]{\cong} & B \end{array}$$

It follows that the map $P(P(E),\alpha) \to P(P(X),p\circ\alpha)$ of based path spaces induced by p is a homeomorphism. ∎

4. Comparison to fibrations and Dydak's Problem. The following lemma is proven by comparing Definition 3.1 with Lemma 2.3 and [35, 2.7.8]

Lemma 4.1. *A Hurewicz fibration has totally path-disconnected fibers if and only if it has the continuous path-covering property.*

Proof of Theorem 1.1. (1)⇒(2) follows from Lemma 4.1.

For (2)⇒(3), suppose $p : E \to X$ has the continuous path-covering property. By Proposition 3.3, it suffices to show that p has the homotopy lifting property with respect to I^n for $n \geq 1$. Note that $p : E \to X$ has the homotopy lifting property with respect to a locally compact Hausdorff space Z if and only if for every map $f : Z \to E$, the induced map $P(p^Z) : P(E^Z, f) \to P(X^Z, p\circ f)$ is surjective. By inductively applying Theorem 3.15 with the exponential homeomorphism $(W^{I^n})^I \cong W^{I^{n+1}}$, we see that for every $n \geq 1$ and map $f : I^n \to E$, the map $P(p^{I^n}) : P(E^{I^n}, f) \to P(X^{I^n}, p\circ f)$ is a homeomorphism. Therefore, p has the homotopy lifting property with respect to all cubes I^n, $n \geq 0$, and is a Serre fibration.

(3)⇒(4) follows from Lemma 2.3. ∎

Since every covering projection is a Hurewicz fibration with discrete fibers [35, Theorem 2.2.3], we have the following.

Corollary 4.2. *Every covering projection has the continuous path-covering property.*

The previous corollary allows us to answer the question: when do we know that a map with the continuous path-covering property is, in fact, a genuine covering projection? Our answer provides a generalization of the classical result [35, Theorem 2.5.10].

COROLLARY 4.3. *Suppose X is locally path-connected and semilocally simply connected and $p : E \to X$ is a map with the continuous path-covering property. If $\mathrm{ev}_1 : P(E, e_0) \to E$ is quotient (for instance, if E is locally path-connected), then p is a covering projection.*

Proof. The hypotheses on X imply that there exists a covering projection $q : E' \to X$ and $e'_0 \in E'$ such that $q_{\#}(\pi_1(E', e'_0)) = p_{\#}(\pi_1(E, e_0))$. By Corollary 4.2, q has the continuous path-covering property. Since $\mathrm{ev}_1 : P(E, e_0) \to E$ is quotient, Corollary 3.13 applies to give a homeomorphism $f : E \to E'$ such that $q \circ f = p$. It follows that p is a covering projection. ■

EXAMPLE 4.4. By considering two one-dimensional planar sets, we construct a counterexample to the converse of (1)⇒(2) in Theorem 1.1. Define

$$A = \{(x, -\sqrt{x - x^2}) \in \mathbb{R}^2 \mid x \in [0,1]\} \cup \bigcup_{n \in \mathbb{N}} \{(t, t/n) \in \mathbb{R}^2 \mid 0 \leq t \leq 2\}.$$

Let $X_1 = A \cup \{(x, \frac{1-x}{2}) \mid 1 \leq x \leq 2\}$ and $X_2 = A \cup [1,2] \times \{0\}$. Define $p : X_1 \to X_2$ to be the identity on A and $p(x, \frac{1-x}{2}) = (x, 0)$, $1 \leq x \leq 2$, on the additional line segment (see Figure 1). Notice that p is a continuous bijection with the continuous path-covering property. However, f is not a fibration since it does not have the homotopy lifting property with respect to the convergent sequence space $S = \{0\} \cup \{1/n \mid n \in \mathbb{N}\}$. In particular, we have $f : S \to X_1$ given by $f(0) = (1, 0)$ and $f(1/n) = (1, 1/n)$. If $g : S \times I \to X_2$ is defined by $g(0, t) = (0, t + 1)$ and $g(1/n, t) = (t + 1, \frac{t+1}{n})$, then we have $p \circ f(s) = g(s, 0)$; however, there is no continuous lift $\widetilde{g} : S \times I \to X_1$ such that $p \circ \widetilde{g} = g$.

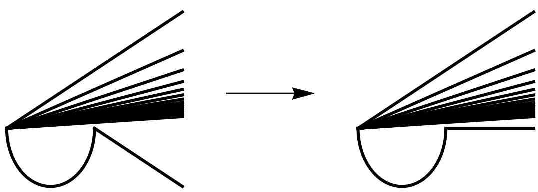


Fig. 1. The map $p : X_1 \to X_2$ has the continuous path-covering property but is not a Hurewicz fibration.

EXAMPLE 4.5 ([24]). Consider the *Infinite Earring Space*

$$\mathbb{E} = \bigcup_{n \in \mathbb{N}} \left\{ (x, y) \in \mathbb{R}^2 \;\middle|\; \left(x - \frac{1}{n}\right)^2 + y^2 = \frac{1}{n^2} \right\}$$

where $\ell_n : S^1 \to \mathbb{E}$ denotes the standard counterclockwise loop around the nth circle based at $b_0 = (0,0)$. Although $\mathbb{E}$ does not admit a simply connected covering space (since it is not semilocally simply connected), it does admit a simply connected space $\widetilde{\mathbb{E}}$ which is a generalized universal covering space in the sense of [24]. The generalized covering map $p : \widetilde{\mathbb{E}} \to \mathbb{E}$ is characterized by its lifting property: given $\widetilde{x} \in \widetilde{\mathbb{E}}$ and a map $f : (Y, y) \to (\mathbb{E}, p(\widetilde{x}))$ from a path-connected, locally path-connected space Y, there exists a unique lift $\widetilde{f} : (Y, y) \to (\widetilde{\mathbb{E}}, \widetilde{x})$ (satisfying $p \circ \widetilde{f} = f$) if and only if $f_{\#}(\pi_1(Y, y)) = 1$. It follows directly from this lifting criterion that p is a Serre fibration with unique path lifting. Moreover, as observed in [24, Example 4.15], distinct fibers of p may not be homeomorphic. Hence, p is not a Hurewicz fibration. We observe that, in fact, p does not have the continuous path-covering property, and therefore provides a counterexample to the converse of (2)⇒(3) in Theorem 1.1.

As noted in [24], $\widetilde{\mathbb{E}} = P(\mathbb{E}, b_0)/\sim$ may be constructed as the set of path-homotopy classes $[\alpha]$ of paths $\alpha \in P(\mathbb{E}, b_0)$. A basic open neighborhood of $[\alpha]$ is of the form $B([\alpha], U) = \{[\alpha \cdot \delta] \mid \delta(I) \subseteq U\}$ where U is an open neighborhood of $\alpha(1)$. The map $p : \widetilde{\mathbb{E}} \to \mathbb{E}$ is the endpoint projection, $p([\alpha]) = \alpha(1)$. In particular, if we take the class $[c_{b_0}]$ of the constant path as our basepoint in $\widetilde{\mathbb{E}}$, then the unique lift $\widetilde{\alpha} : (I, 0) \to (\widetilde{\mathbb{E}}, [c_{b_0}])$ ends at $[\alpha]$.

Notice that the sequence $\alpha_n = \ell_n \cdot \ell_1$ converges to ℓ_1 in $P(\mathbb{E}, b_0)$. If p had the continuous path-covering property, then the sequence of lifts $\widetilde{\alpha}_n : (I, 0) \to (\widetilde{\mathbb{E}}, b_0)$ would converge to the lift $\widetilde{\ell}_1$. In particular, the sequence of endpoints $\widetilde{\alpha}_n(1) = [\alpha_n]$ would converge to $\widetilde{\ell}_1(1) = [\ell_1]$ in $\widetilde{\mathbb{E}}$. However, if U is any neighborhood of b_0 not containing the first circle of $\mathbb{E}$, then the basic open neighborhood $B([\ell_1], U)$ of $[\ell_1]$ only contains homotopy classes of paths that begin with $[\ell_1]$. Hence $[\alpha_n] \notin B([\ell_1], U)$ for any $n \in \mathbb{N}$, a contradiction. We conclude that p does not have the continuous path-covering property.

Let D^2 denote the closed unit disk with basepoint $d_0 = (1, 0)$.

Problem 4.6 (Dydak's Unique Lifting Problem [16, Problem 2.3]). *If E is a connected, locally path-connected space and $p : E \to D^2$ is a map with the path-covering property, must p be a homeomorphism?*

Dydak's Unique Lifting Problem is curiously difficult. We identify an interesting connection between this problem and Theorem 1.1. To do so, we recall a functorial construction that "locally path-connectifies" spaces. Given a space X, the *locally path-connected coreflection* of X is the space $\mathrm{lpc}(X)$ with the same underlying set as X but whose topology is generated by the basis consisting of all path components of the open sets in X. This new topology on X is generally finer than the original topology, i.e. the identity function $\mathrm{id} : \mathrm{lpc}(X) \to X$ is continuous. Moreover, $\mathrm{lpc}(X)$ is characterized by its universal property: if $f : Y \to X$ is a map from a locally path-connected space Y, then $f : Y \to \mathrm{lpc}(X)$ is continuous with respect to the topology of $\mathrm{lpc}(X)$. An immediate consequence of this fact is the equality $X^Y = \mathrm{lpc}(X)^Y$ of mapping sets if Y is locally path-connected. In particular, X and $\mathrm{lpc}(X)$ share the same set of paths and path-homotopies. Hence, $\mathrm{id} : \mathrm{lpc}(X) \to X$ is a bijective weak homotopy equivalence, which sometimes is even a Hurewicz fibration [35, Example 2.4.8].

DEFINITION 4.7. We say a map $p : E \to X$ has the *disk-covering property* if for every $e \in E$, the induced function $F : (E, e)^{(D^2,d_0)} \to (X, p(e))^{(D^2,d_0)}$ is bijective.

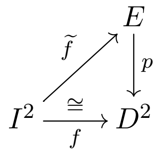
LEMMA 4.8. *For any map $p : E \to X$, the following are equivalent:*

(1) *p has the disk-covering property,*
(2) *p has the path-covering property and the homotopy lifting property with respect to I,*
(3) *$p : E \to X$ is a Serre fibration with totally path-disconnected fibers,*
(4) *p has totally path-disconnected fibers and the homotopy lifting property with respect to all first countable, locally path-connected, simply connected spaces.*

Proof. The directions (4)⇒(3)⇒(2)⇒(1) are clear. Additionally, (1)⇒(2) follows directly from the fact that $(I^2, I \times \{0\})$ has the homotopy extension property and $I^2/I\times\{0\} \cong D^2$. To prove (2)⇒(4), suppose $p : E \to X$ has the path-covering property and the homotopy lifting property with respect to I. The fibers of p are totally path-disconnected by Proposition 3.3. Note that Lemma 2.4 applies to p. Let Z be a first countable, locally path-connected, and simply connected space, and $f : Z \to E$ and $g : Z \times I \to X$ be maps such that $p \circ f(z) = g(z, 0)$ for all $z \in Z$. Fix $z_0 \in Z$. Set $e_0 = f(z_0)$ and $x_0 = p(e_0)$. Since p has unique lifting of all paths and path-homotopies and since $Z \times I$ is simply connected, there is a unique function $\widetilde{g} : (Z \times I, (z_0, 0)) \to (E, e_0)$ such that $p \circ \widetilde{g} = g$ and defined so that if $(z, t) \in Z \times I$ and γ is a path from $(z_0, 0)$ to (z, t), then $\widetilde{g}(z, t)$ is the endpoint of the lift $\widetilde{g \circ \gamma} \in P(E, e_0)$. It suffices to check that $\widetilde{g}$ is continuous. Consider a convergent sequence $\{(z_m, t_m)\} \to (z, t)$ in $Z \times I$. Since $Z \times I$ is first countable and locally path-connected, there is a path $\gamma \in P(Z \times I, (z_0, 0))$ such that $\gamma(1/2) = (z, t)$ and if $s_m = \frac{1}{2} + \frac{1}{m+1}$, then $\gamma(s_m) = (z_m, t_m)$. Consider the lift $\widetilde{g \circ f} : (I, 0) \to (E, e_0)$. Applying the uniqueness of path lifting and the well-definedness of $\widetilde{g}$, we see that $\widetilde{g \circ \gamma}(1/2) = \widetilde{g}(z, t)$ and $\widetilde{g \circ \gamma}(s_m) = \widetilde{g}(z_m, t_m)$ for all $m \in \mathbb{N}$. The continuity of $\widetilde{g \circ \gamma}$ applied to $\{t_m\} \to 1/2$ gives $\widetilde{g}(z_m, t_m) \to \widetilde{g}(z, t)$. Thus $\widetilde{g}$ is continuous. ∎

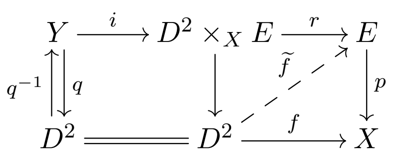
THEOREM 4.9. *Dydak's Unique Lifting Problem has a positive answer if and only if properties* (3) *and* (4) *in Theorem* 1.1 *are equivalent for all maps.*

Proof. First, suppose (3) and (4) in Theorem 1.1 are equivalent for all maps. If E is a path-connected, locally path-connected space and $p : E \to D^2$ is a map with the path-covering property, then, by assumption, p is a Serre fibration. Recall that $d_0 = (1, 0) \in D^2$ and let $e_0 \in p^{-1}(d_0)$. Let $\alpha : (I, 0) \to (D^2, d_0)$, $\alpha(t) = (\cos(\pi t), -\sin(\pi t))$ be the arc on the boundary of the lower semicircle and $f : (I^2, \mathbf{0}) \to (D^2, d_0)$ be a homeomorphism such that $f(t, 0) = \alpha(t)$. Since p is a Serre fibration, there exists a map $\widetilde{f} : (I^2, \mathbf{0}) \to (E, e_0)$ satisfying $p \circ \widetilde{f} = f$. Clearly, we have $p \circ (\widetilde{f} \circ f^{-1}) = \mathrm{id}_{D^2}$:



Let $e \in E$ and find a path $\widetilde{\beta} : I \to E$ from e_0 to e. Let $\beta = p \circ \widetilde{\beta}$. Now $\widetilde{f} \circ f^{-1} \circ \beta : (I, 0) \to (E, e_0)$ is a path satisfying $p \circ \widetilde{f} \circ f^{-1} \circ \beta = \beta$ and thus $\widetilde{f} \circ f^{-1} \circ \beta = \widetilde{\beta}$. In particular, $\widetilde{f} \circ f^{-1} \circ p(e) = \widetilde{f} \circ f^{-1} \circ \beta(1) = \widetilde{\beta}(1) = e$. Thus $\widetilde{f} \circ f^{-1} \circ p = \mathrm{id}_E$, proving that p is a homeomorphism with inverse $\widetilde{f} \circ f^{-1}$.

Next, we suppose Dydak's Unique Lifting Problem has a positive answer and that $p : E \to X$ is a map with the path-covering property. By Lemma 4.8, it suffices to show that p has the disk-covering property. Let $e \in E$ and consider the induced map $F : (E, e)^{(D^2, d_0)} \to (X, p(e))^{(D^2, d_0)}$ given by $F(g) = p \circ g$. Since p has unique path lifting, it is clear that F is injective. Let $f : (D^2, d_0) \to (X, p(e_0))$ be a map. Let $D^2 \times_X E = \{(b, e) \mid f(b) = p(e)\}$ be the pullback of f and p topologized as a subspace of $D^2 \times E$. We take Y to be the locally path-connected coreflection of the path component of $D^2 \times_X E$ containing (d_0, e_0). Let $q : Y \to D^2$ be the resulting projection. Since p has the disk-covering property, the universal property of the pullback and the locally path-connected coreflection make it clear that q has the path-covering property. Hence, by assumption, q is a homeomorphism. It follows that if $i : Y \to D^2 \times_X E$ is the canonical continuous inclusion (though it may not be an embedding) and $r : D^2 \times_X E \to E$ is the projection, then $\widetilde{f} = r \circ i \circ q^{-1} : (D^2, d_0) \to (E, e_0)$ is a map such that $p \circ \widetilde{f} = f$. We conclude that F is onto. Thus p has the disk-covering property.



∎

If Dydak's Problem has an affirmative answer, the following corollary becomes a striking consequence; the conclusion implies that, within standard scenarios, only the unique lifting of paths is required to guarantee the strongest kind of structure, namely, that of a genuine covering projection. For instance, it would imply that any map $p : E \to X$ of connected manifolds with the path-covering property must be a covering projection.

COROLLARY 4.10. *Suppose Dydak's Unique Path-Lifting Problem has an affirmative answer. If $p : E \to X$ has the path-covering property where X is first countable, locally path-connected, semilocally simply connected, and $\mathrm{ev}_1 : P(E, e_0) \to E$ is quotient, then p is a covering projection.*

Proof. A map p as in the hypotheses is a Serre fibration with totally path-disconnected fibers by Theorem 4.9. Let $H = p_{\#}(\pi_1(E, e_0))$. Given the conditions on X, there exists a covering projection $q : (E', e'_0) \to (X, x_0)$ such that $q_{\#}(\pi_1(E', e'_0)) = H$. Since q has the continuous path-covering property, by Lemma 3.12, there exists a unique continuous lift $\widetilde{p} : (E, e_0) \to (E', e'_0)$ such that $q \circ \widetilde{p} = p$. Since p and q are Serre fibrations with the unique path-lifting property satisfying $q_{\#}(\pi_1(E', e'_0)) = p_{\#}(\pi_1(E, e_0))$, $\widetilde{p}$ is a bijection with the path-covering property.

$$\begin{array}{ccc} P(E, e_0) & \xrightarrow{P(\widetilde{p})} & P(E', e'_0) \\ {\scriptstyle \mathrm{ev}_1}\downarrow & & \downarrow{\scriptstyle \mathrm{ev}_1} \\ E & \xrightarrow{\widetilde{p}} & E' \end{array}$$

Since X is first countable, so is the covering space E'. Let $\{e'_m\} \to e'$ be a convergent sequence in E', let $\widetilde{p}(e_m) = e'_m$ and $\widetilde{p}(e) = e'$. Since E' is first countable and locally path-connected, there is a path $\widetilde{\alpha} \in P(E', e'_0)$ such that $\widetilde{\alpha}(1/2) = e'$ and $\widetilde{\alpha}(t_m) = e'_m$ where $t_m = \frac{1}{2} + \frac{1}{m+1}$, $m \in \mathbb{N}$. Since $\widetilde{p}$ has the path-covering property, there is a unique path $\widetilde{\beta} \in P(E, e_0)$ such that $\widetilde{p} \circ \widetilde{\beta} = \widetilde{\alpha}$. Since $\widetilde{\beta}(t_m) = e_m$ and $\widetilde{\beta}(1/2) = e$, the continuity of $\widetilde{\beta}$ gives $\{e_m\} \to e$. This proves that the inverse of $\widetilde{p}$ is continuous. Since $\widetilde{p}$ is a homeomorphism and q is a covering map, p is a covering map as well. ∎

5. Inducing embeddings on topologized fundamental groups. Recall from the introduction the basic properties of the quotient topology on the homotopy groups. A key feature of this topology is that a convergent net $\{\alpha_j\}_{j \in J} \to \alpha$ of based loops in $\Omega(X, x_0)$ gives rise to a convergent net of homotopy classes $\{[\alpha_j]\}_{j \in J} \to [\alpha]$ in $\pi_1(X, x_0)$.

LEMMA 5.1. *If $\{e\}$ is closed in E and $p : E \to X$ has the continuous path-covering property, then the induced map $\Omega^n(p) : \Omega^n(E, e) \to \Omega^n(X, p(e))$ is a closed embedding for $n = 1$ and a homeomorphism for $n \geq 2$.*

Proof. For the case $n = 1$, note that since $\{e\}$ is closed in E, $\mathrm{ev}_1^{-1}(e) = \Omega(E, e)$ is a closed subspace of $P(E, e)$. By assumption, p induces a homeomorphism $P(p) : P(E, e) \to P(X, p(e))$ on path spaces, which restricts to an embedding $\Omega(p) : \Omega(E, e) \to \Omega(X, p(e))$. Recall that a path $\beta \in \Omega(X, p(e))$ lies in the image of $\Omega(p)$ if and only if the lift $\widetilde{\beta} \in P(E, e)$ ends at e, i.e. is a loop. Therefore, it suffices to show that $\Omega(p)$ has closed image in $\Omega(X, p(e))$. Suppose, to obtain a contradiction, that $\Omega(p)$ does not have closed image. Then there is a convergent net $\{\beta_j\}_{j\in J} \to \beta_\infty$ in $\Omega(X, p(e))$ where $\beta_j \in \mathrm{Im}(\Omega(p))$ for every $j \in J$ and $\beta_\infty \notin \mathrm{Im}(\Omega(p))$. This net uniquely defines a map $g : (F(J), v_0) \to (X, p(e))$ where $g(j, t) = \beta_j(t)$ for $j \in J \cup \{\infty\}$. According to Remark 3.8, there is a unique map $\widetilde{g} : (F(J), v_0) \to (E, e)$ such that $p \circ \widetilde{g} = g$. Let $\widetilde{\beta}_j$ be the path $\widetilde{\beta}_j(t) = \widetilde{g}(j, t)$ for $j \in J \cup \{\infty\}$. Then $\{\widetilde{\beta}_j\}_{j\in J} \to \widetilde{\beta}_\infty$ in $P(E, e)$. But since β_j lies in the image of $\Omega(p)$ for each $j \in J$, we have $\widetilde{\beta}_j \in \Omega(E, e)$ for all $j \in J$. Additionally, since β_∞ does not lie in the image of $\Omega(p)$, β_∞ does not lift to a loop, i.e. $\widetilde{\beta}_\infty \notin \Omega(E, e)$. However, this contradicts the fact that $\Omega(E, e)$ is closed in $P(E, e)$.

For $n \geq 2$, Corollary 3.10 ensures that $p^{(I^n,\mathbf{0})} : (E, e)^{(I^n,\mathbf{0})} \to (X, p(e))^{(I^n,\mathbf{0})}$ is a homeomorphism. Hence the restriction $\Omega^n(p) : \Omega^n(E, e) \to \Omega^n(X, p(e))$ is an embedding. We check that $\Omega^n(p)$ is onto. Let $f : (I^n, \partial I^n) \to (X, p(e))$ be a map and consider the unique lift $\widetilde{f} : (I^n, \mathbf{0}) \to (E, e)$. It suffices to check that $\widetilde{f}(\partial I^n) = e$. Let $\mathbf{x} \in \partial I^n$ and $\gamma : (I, 0) \to (I^n, \mathbf{0})$ be the linear path from $\mathbf{0}$ to $\mathbf{x}$. Hence, we have a loop $f \circ \gamma : (I, \{0, 1\}) \to (X, p(e_2))$, which factors through the simply connected space $S^n \cong I^n/\partial I^n$. Since $[f \circ \gamma] = 1 \in \pi_1(X, p(e))$, Lemma 2.4 guarantees that the unique lift $\widetilde{f \circ \gamma} \in P(E, e)$ is a loop. Since $(p \circ \widetilde{f}) \circ \gamma = f \circ \gamma = p \circ \widetilde{f \circ \gamma}$, unique path lifting gives $\widetilde{f} \circ \gamma = \widetilde{f \circ \gamma}$. Hence $\widetilde{f}(\mathbf{x}) = \widetilde{f}(\gamma(1)) = \widetilde{f \circ \gamma}(1) = e$. This proves $\widetilde{f}(\partial I^n) = e$. ∎

Theorem 5.2. *If $\{e\}$ is closed in E and $p : E \to X$ has the continuous path-covering property, then the induced homomorphism $p_\# : \pi_n(E, e) \to \pi_n(X, p(e))$ is a closed embedding for $n = 1$ and an isomorphism of quasitopological groups for $n \geq 2$.*

Proof. For $n = 1$, $p_\#$ is injective by Lemma 2.4 and continuous by the functoriality of the quotient topology. Consider the following commutative diagram where the vertical maps are the natural quotient maps identifying homotopy classes:

$$\begin{array}{ccc} \Omega(E, e) & \xrightarrow{P(p)} & \Omega(X, p(e)) \\ {\scriptstyle q_E}\downarrow & & \downarrow{\scriptstyle q_X} \\ \pi_1(E, e) & \xrightarrow{p_\#} & \pi_1(X, p(e)) \end{array}$$

Suppose that $C \subseteq \pi_1(E, e)$ is closed and non-empty. Then $q_E^{-1}(C)$ is closed in $\Omega(E, e)$ and, since the top map is a closed embedding by Lemma 5.1, $P(p)(q_E^{-1}(C))$ is closed in $\Omega(X, p(e))$. Since q_X is a quotient map, it suffices to show that $q_X^{-1}(p_\#(C))$ is closed. However, this is a consequence of the equality $P(p)(q_E^{-1}(C)) = q_X^{-1}(p_\#(C))$, which may be verified using standard unique lifting arguments.

For $n \geq 2$, consider the following commuting diagram where the vertical maps are the natural quotient maps:

$$\begin{array}{ccc} \Omega^n(E,e) & \xrightarrow{\Omega^n(p)} & \Omega^n(X,p(e)) \\ {\scriptstyle q_E}\downarrow & & \downarrow{\scriptstyle q_X} \\ \pi_n(E,e) & \xrightarrow{p_\#} & \pi_n(X,p(e)) \end{array}$$

By Lemma 5.1, the top map is a homeomorphism. The universal property of quotient maps ensures that $p_\#$ is a topological quotient map. By Theorem 1.1, p is a Serre fibration with totally path-disconnected fibers. The injectivity of $p_\#$ follows from the long exact sequence of homotopy groups associated with p. Thus, $p_\#$ is a group isomorphism and a homeomorphism. ∎

COROLLARY 5.3. *Suppose $p_i : E_i \to X$, $i \in \{1,2\}$, are maps of T_1 spaces with the continuous path-covering property and $f : E_1 \to E_2$ is a weak homotopy equivalence such that $p_2 \circ f = p_1$. Then f is a weak topological homotopy equivalence.*

Proof. First, notice that f must be a bijection. By Lemma 3.5, f has the continuous path-covering property and it follows from Theorem 5.2 that f is a weak topological homotopy equivalence. ∎

THEOREM 5.4. *Suppose $p : (E,e_0) \to (X,x_0)$ has the continuous path-covering property and suppose $H = p_\#(\pi_1(E,e_0))$ where $\{x_0\}$ is closed in X. Then there is a canonical continuous bijection $\phi : \pi_1(X,x_0)/H \to p^{-1}(x_0)$ defined by $\phi(H[\alpha]) = \widetilde{\alpha}(1)$ where $\widetilde{\alpha}$ is the unique lift of α starting at e_0. Moreover, if $\mathrm{ev}_1 : P(E,e_0) \to E$ is a quotient map, then ϕ is a homeomorphism.*

Proof. The map ϕ is analogous to the correspondence used in classical covering space theory and is well-defined in view of Lemma 2.4. To verify the continuity of ϕ, we consider the following diagram where i is inclusion and the vertical map $q(\alpha) = H[\alpha]$ is quotient:

$$\begin{array}{ccc} \Omega(X,x_0) & \xrightarrow{\quad i \quad} & P(X,x_0) \\ & & \cong\ \downarrow{\scriptstyle P(p)^{-1}} \\ {\scriptstyle q}\downarrow & \searrow^{\psi} & P(E,e_0) \\ & & \downarrow{\scriptstyle \mathrm{ev}_1} \\ \pi_1(X,x_0)/H & \overset{\phi}{\dashrightarrow} & E \end{array}$$

The composition $\psi = \mathrm{ev}_1 \circ P(p)^{-1} \circ i$ is continuous, and it maps a loop α based at x to the endpoint $\widetilde{\alpha}(1)$ of the lift starting at e_0. Since E is assumed to be path-connected, ψ maps $\Omega(X, x_0)$ onto the fiber $p^{-1}(x_0)$. Lemma 2.4 implies that $\psi(\alpha) = \psi(\beta)$ if and only if $H[\alpha] = H[\beta]$. Since q is quotient, we have a unique continuous bijection $\phi : \pi_1(X, x_0)/H \to p^{-1}(x_0)$ defined by $\phi(H[\alpha]) = \psi(\alpha)$.

If ev_1 is quotient, then $P(p)^{-1} \circ \mathrm{ev}_1$ is quotient. Since $\{x_0\}$ is closed, $p^{-1}(x_0)$ is closed in E. We have $(P(p)^{-1} \circ \mathrm{ev}_1)^{-1}(p^{-1}(x_0)) = \Omega(X, x_0)$ and thus the restriction $\psi : \Omega(X, x_0) \to p^{-1}(x_0)$ of the quotient map $P(p)^{-1} \circ \mathrm{ev}_1$ is also quotient. Since $\phi \circ q = \psi$ where q and ψ are quotient, ϕ must also be quotient and thus a homeomorphism. ∎

Remark 5.5. The converse of Theorem 5.4 is false. Indeed, if $X = E$ is any simply connected space for which ev_1 is not quotient (e.g. X_1 or X_2 from Example 4.4) and $p = \mathrm{id}_X$, then ϕ is a homeomorphism of 1-point spaces.

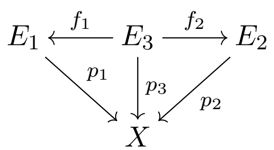
Corollary 5.6. *If $p : (E, e_0) \to (X, x_0)$ has the continuous path-covering property and $H = p_\#(\pi_1(E, e_0))$, then the coset space $\pi_1(X, x_0)/H$ is totally path-disconnected.*

Proof. By Proposition 3.3, $p^{-1}(x_0)$ is totally path-disconnected and the proof of Theorem 5.4 implies that $\pi_1(X, x_0)/H$ continuously injects into $p^{-1}(x_0)$. ∎

6. Classifying maps with continuous path-covering property. To make the statement of Theorem 1.2 precise, we define the following relations.

Definition 6.1. Consider two maps $p_1 : E_1 \to X$ and $p_2 : E_2 \to X$ with the continuous path-covering property.

(1) We say p_1 and p_2 are *equivalent* if there exists a homeomorphism $h : E_1 \to E_2$ such that $p_2 \circ h = p_1$, and we refer to h as an *equivalence.*
(2) A *simple weak equivalence between* p_1 *and* p_2 is a triple (p_3, f_1, f_2) where $p_3 : E_3 \to X$ has the continuous path-covering property and $f_i : E_3 \to E_i$, $i \in \{1, 2\}$, are weak homotopy equivalences such that $p_i \circ f_i = p_3$. If a simple weak equivalence exists between p_1 and p_2, we write $p_1 \sim_s p_2$.



(3) We say p_1 and p_2 are *weakly equivalent* if there exists a finite chain of simple weak equivalences $p_1 = q_1 \sim_s \cdots \sim_s q_m = p_2$. If such a chain exists, we say p_1 and p_2 are *weakly equivalent.*

REMARK 6.2. Note that if we have $p_2 \circ f = p_1$ for maps $p_i : E_i \to X$ with the continuous path-covering property, then f has the continuous path-covering property by Lemma 3.5 and f must be surjective. If, in addition, f is π_1-surjective, then standard lifting arguments show that f is a bijection. If f induces an isomorphism on π_1, then f must also be a weak topological homotopy equivalence by Theorem 5.2 (if X is at least T_1, then E is T_1 by Proposition 3.3). Hence, one may equivalently define "simple weak equivalence" by replacing weak homotopy equivalences f_1, f_2 with either the weaker notion of π_1-surjective maps or the stronger notion of bijective, weak topological homotopy equivalences.

LEMMA 6.3. *If $p_1 : E_1 \to X$ and $p_2 : E_2 \to X$ are weakly equivalent maps with the continuous path-covering property, and $e_1 \in E_1$, then $(p_1)_\#(\pi_1(E_1, e_1)) = (p_2)_\#(\pi_1(E_2, e_2))$ for some $e_2 \in E_2$.*

Proof. Fix $e_1 \in E_1$ and $x_0 = p_1(e_1)$. If there is a weak homotopy equivalence $f : E_1 \to E_2$ such that $p_2 \circ f = p_1$, we set $e_2 = f(e_1)$. Since f is π_1-surjective, the equality $(p_1)_\#(\pi_1(E_1, e_1)) = (p_2)_\#(\pi_1(E_2, e_2))$ follows. If there is a weak homotopy equivalence $g : E_2 \to E_1$ such that $p_1 \circ g = p_2$, then g is surjective by Remark 6.2. Hence, we may find $e_2 \in E_2$ with $g(e_2) = e_1$, from which $(p_1)_\#(\pi_1(E_1, e_1)) = (p_2)_\#(\pi_1(E_2, e_2))$ follows. Applying zig-zags of simple weak equivalences now gives the lemma. ∎

Notice that Lemma 6.3 implies that the weak equivalence classes of maps with the continuous path-covering property over a given space X form a set. To give a self-contained proof of Theorem 1.2, we require a sequence of lemmas involving quotient space constructions.

LEMMA 6.4. *Suppose X is a path-connected Hausdorff space, $x_0 \in X$, and $c(X)$ is the space with the same underlying set as X but with the quotient topology inherited from $\mathrm{ev}_1 : P(X, x_0) \to X$. Then*

(1) *the identity function $f : c(X) \to X$ has the continuous path-covering property and is a weak topological homotopy equivalence,*
(2) $\mathrm{ev}_1 : P(c(X), x_0) \to c(X)$ *is quotient, i.e. $c(c(X)) = c(X)$.*

Proof. Let J be a directed set and $g : (F(J), v_0) \to (X, x_0)$ be a based map on the directed arc-fan over J. Applying Remark 3.8, we show that $g : (F(J), v_0) \to (c(X), x_0)$ is continuous. Since $F(J)$ is contractible, $\mathrm{ev}_1 : P(F(J), v_0) \to F(J)$ is a retraction and is therefore a topological quotient map. Consider the induced map $P(g)$ in the diagram below:

$$\begin{array}{ccc} P(F(J), v_0) & \xrightarrow{P(g)} & P(X, x_0) \\ \Big\downarrow{\scriptstyle \mathrm{ev}_1} & & \Big\downarrow{\scriptstyle \mathrm{ev}_1} \\ F(J) & \xrightarrow{\quad g \quad} & c(X) \end{array}$$

Since the top composition is continuous and the left vertical map is quotient, the bottom map is continuous by the universal property of quotient maps. Therefore, since all maps of directed arc-fans lift, f has the continuous path-covering property. Having a finer topology than X, $c(X)$ is Hausdorff. Theorem 5.2 then shows that f induces a closed embedding on π_1 and a topological isomorphism on π_n for $n \geq 2$. Hence, it suffices to show f is π_1-surjective. Given a loop $\alpha \in \Omega(X, x_0)$, there is a unique lift $\widetilde{\alpha} \in P(c(X), x_0)$ such that $f \circ \widetilde{\alpha} = \alpha$. However, since the underlying function of f is the identity, it must be that $\widetilde{\alpha} = \alpha$ as a loop. Since $\Omega(f) : \Omega(c(X), x_0) \to \Omega(X, x_0)$ is a bijection, it follows that $f_{\#} : \pi_1(c(X), x_0) \to \pi_1(X, x_0)$ is surjective. This completes the proof of (1).

For (2), recall that we have shown f has the continuous path-covering property. Hence, as the top map in the triangle below is a homeomorphism and the right map is quotient by construction, the left evaluation map is the composition of quotient maps and is therefore quotient:

$$\begin{array}{ccc} P(c(X), x_0) & \xrightarrow{P(f)} & P(X, x_0) \\ & {\scriptstyle \mathrm{ev}_1} \searrow \quad \swarrow {\scriptstyle \mathrm{ev}_1} & \\ & c(X) & \end{array}$$

■

Example 6.5. Path-connected spaces (Z, z_0) for which $c(Z) \to Z$ is not a homeomorphism exist, e.g. the spaces X_1 and X_2 in Example 4.4, and Zeeman's example [31, Example 6.6.14]. For any such space, the identity function $c(Z) \to Z$ and the identity map $Z \to Z$ are non-equivalent, weakly equivalent maps with the continuous path-covering property that both correspond to $H = \pi_1(Z, z_0)$. Therefore, the main statement of Theorem 1.2 only holds using our notion of weak equivalence.

We use the construction of $c(X)$ to prove the converse of Lemma 6.3.

Lemma 6.6. *Let $p_1 : E_1 \to X$ and $p_2 : E_2 \to X$ be maps with the continuous path-covering property. Then the following are equivalent:*

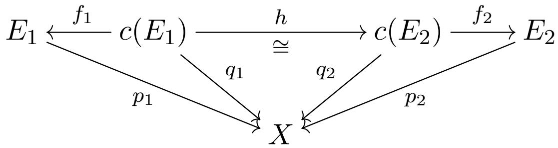
(1) *p_1 and p_2 are weakly equivalent,*
(2) *for every $e_1 \in E_1$, we have $(p_1)_{\#}(\pi_1(E_1, e_1)) = (p_2)_{\#}(\pi_1(E_2, e_2))$ for some $e_2 \in E_2$,*
(3) *for all $e_1 \in E_1$ and $e_2 \in E_2$ such that $p_1(e_1) = x_0 = p_2(e_2)$, the subgroups $(p_1)_{\#}(\pi_1(E_1, e_1))$ and $(p_2)_{\#}(\pi_1(E_2, e_2))$ are conjugate in $\pi_1(X, x_0)$.*

Moreover, if $\mathrm{ev}_1 : P(E_1, e_1) \to E_1$ and $\mathrm{ev}_1 : P(E_2, e_2) \to E_2$ are quotient, then "weak equivalence" may be replaced by "equivalence".

Proof. (1)⇒(2) is Lemma 6.3 and (2)⇔(3) follows from standard covering space theory arguments. To prove (2)⇒(1), suppose that

$$(p_1)_{\#}(\pi_1(E_1, e_1)) = (p_2)_{\#}(\pi_1(E_2, e_2))$$

for some $e_1 \in E_1$ and $e_2 \in E_2$. Let $c(E_1)$ and $c(E_2)$ be the spaces constructed as in Lemma 6.4 so that the continuous identity functions $f_1 : c(E_1) \to E_1$ and $f_2 : c(E_2) \to E_2$ have the continuous path-covering property and are weak topological homotopy equivalences. For $i \in \{1, 2\}$, let $q_i : c(E_i) \to X$ be the map $q_i = p_i \circ f_i$. Since q_i is the composition of maps with the continuous path-covering property, q_i also has the continuous path-covering property (recall Lemma 3.5). Moreover, since f_1 and f_2 are weak homotopy equivalences, we have $(q_1)_{\#}(\pi_1(c(E_1), e_1)) = (q_2)_{\#}(\pi_1(c(E_2), e_2))$. By Lemma 6.4(2), $\mathrm{ev}_1 : P(c(E_i), e_i) \to c(E_i)$ is quotient for $i \in \{1, 2\}$. Therefore, Corollary 3.13 applies to give a homeomorphism $h : (c(E_1), e_1) \to (c(E_2), e_2)$ such that $q_2 \circ h = q_1$.



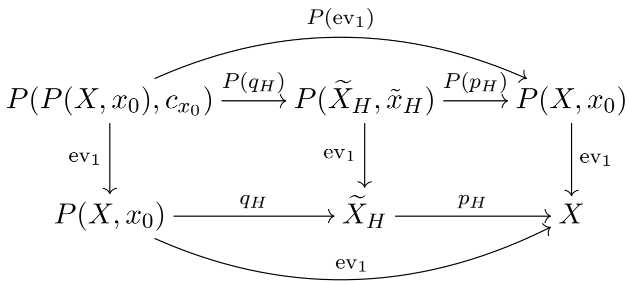
For the final statement of the lemma, suppose $\mathrm{ev}_1 : P(E_i, e_i) \to E_i$ is quotient for $i \in \{1, 2\}$. Then f_1 and f_2 are true identity maps and thus homeomorphisms. It follows that p_1 and p_2 are equivalent. ∎

The previous lemma settles the uniqueness claims in Theorem 1.2. We now focus on existence. Fix a based space (X, x_0), a subgroup $H \leq \pi_1(X, x_0)$, and let $\widetilde{X}_H = P(X, x_0)/{\sim}$ be the quotient space where $\alpha \sim \beta$ if and only if $\alpha(1) = \beta(1)$ and $[\alpha \cdot \beta^-] \in H$. Let $H[\alpha]$ denote the equivalence class of $\alpha \in P(X, x_0)$ and let $q_H : P(X, x_0) \to \widetilde{X}_H$, $q_H(\alpha) = H[\alpha]$, be the quotient map. We write $\tilde{x}_H$ to represent $H[c_{x_0}]$, which we take to be the basepoint of $\widetilde{X}_H$. Let $p_H : \widetilde{X}_H \to X$, $p_H(H[\alpha]) = \alpha(1)$ be the endpoint evaluation map.

Lemma 6.7. *For any path-connected space* (X, x_0),

(1) $\mathrm{ev}_1 : P(\widetilde{X}_H, \tilde{x}_H) \to \widetilde{X}_H$ *is a quotient map,*
(2) $P(p_H) : P(\widetilde{X}_H, \tilde{x}_H) \to P(X, x_0)$ *is a retraction.*

Proof. Consider the following commutative diagram:



Since $P(X,x_0)$ is contractible in a canonical way, we may define a map $\mathscr{S} : P(X,x_0) \to P(P(X,x_0),c_{x_0})$ by setting $\mathscr{S}(\alpha)(t)(s) = \alpha(st)$.

- For the top map $P(\mathrm{ev}_1)$, notice that $P(\mathrm{ev}_1)(\beta) = \mathrm{ev}_1 \circ \beta \in P(X,x_0)$ and thus $(P(\mathrm{ev}_1)(\beta))(t) = \beta(t)(1)$. Therefore,
$$P(\mathrm{ev}_1)(\mathscr{S}(\alpha))(t) = \mathscr{S}(\alpha)(t)(1) = \alpha(t)$$
for all $t \in I$, giving $P(\mathrm{ev}_1) \circ \mathscr{S} = \mathrm{id}_{P(X,x_0)}$.
- For the left vertical map $\mathrm{ev}_1 : P(P(X,x_0),c_{x_0}) \to P(X,x_0)$, we have $\mathscr{S}(\alpha)(1) = \alpha$ and thus $\mathrm{ev}_1 \circ \mathscr{S} = \mathrm{id}_{P(X,x_0)}$.

Hence, $\mathscr{S}$ is a section to both the top map $P(\mathrm{ev}_1)$ and the left vertical map ev_1. In particular, both maps are quotient. In the left square, the composition $q_H \circ \mathrm{ev}_1$ is quotient. It follows that the middle vertical map $\mathrm{ev}_1 : P(\widetilde{X}_H,\tilde{x}_H) \to \widetilde{X}_H$ is quotient, proving (1). In the top triangle, we have $P(p_H) \circ (P(q_H) \circ \mathscr{S}) = P(\mathrm{ev}_1) \circ \mathscr{S} = \mathrm{id}_{P(X,x_0)}$ and thus $P(q_H) \circ \mathscr{S}$ is a section to $P(p_H)$, proving (2). ∎

Remark 6.8. The section $P(q_H) \circ \mathscr{S}$ in the proof of Lemma 6.7 guarantees that for a given $H \le \pi_1(X,x_0)$, every path $\alpha \in P(X,x_0)$ admits a canonical lift $P(q_H) \circ \mathscr{S}(\alpha) = \widetilde{\alpha}_H : I \to \widetilde{X}_H$ of α called the *standard lift* and defined by $\widetilde{\alpha}_H(t) = H[\alpha_t]$ where $\alpha_t(s) = \alpha(st)$ is the linear reparameterization of $\alpha|_{[0,t]}$ and $\alpha_0 = c_{x_0}$.

Proposition 6.9. *The endpoint projection $p_H : \widetilde{X}_H \to X$ is a quotient map if and only if* $\mathrm{ev}_1 : P(X,x_0) \to X$ *is quotient.*

Proof. Since $\mathrm{ev}_1 = p_H \circ q_H$ as maps $P(X,x_0) \to X$ where q_H is quotient, the conclusion follows from basic properties of quotient maps. ∎

For a given path $\alpha \in P(X,x_0)$, consider each of the following pullbacks (with the respective subspace topology):

- $G_\alpha = \{(x,t) \in X \times I \mid \alpha(t) = x\}$, the graph of α,
- $E_{\alpha,H} = \{(H[\beta],t) \in \widetilde{X}_H \times I \mid \beta(1) = \alpha(t)\}$,
- $P_\alpha = \{(\beta,t) \in P(X,x_0) \times I \mid \beta(1) = \alpha(t)\}$.

Recalling that $\alpha_t(s) = \alpha(st)$ for $t \in I$, define $\phi_\alpha : P_\alpha \to \Omega(X,x_0)$ by $\phi_\alpha(\beta,t) = \beta \cdot \alpha_t^-$. Since $t \mapsto \alpha_t$ defines a path in $P(X,x_0)$ and the concatenation $\{(\gamma,\delta) \in P(X,x_0)^2 \mid \gamma(1) = \delta(1)\} \to \Omega(X,x_0)$, $(\gamma,\delta) \mapsto \gamma \cdot \delta^-$, is continuous, ϕ_α is also continuous.

Lemma 6.10. *If a path $\alpha \in P(X,x_0)$ has closed graph $G_\alpha \subseteq X \times I$, then the function $\psi_{\alpha,H} : E_{\alpha,H} \to \pi_1(X,x_0)/H$, $\psi_{\alpha,H}(H[\beta],t) = H[\beta \cdot \alpha_t^-]$, is continuous.*

Proof. First, we observe that $\psi_{\alpha,H}$ is well-defined. Indeed, if $(H[\beta],t) = (H[\gamma],t)$, then $[\beta \cdot \alpha_t^-][\alpha_t \cdot \gamma^-] = [\beta \cdot \gamma^-] \in H$ and thus $H[\beta \cdot \alpha_t^-] = H[\gamma \cdot \alpha_t^-]$.

Let $q_H : P(X,x_0) \to \widetilde{X}_H$ and $\pi_H : \Omega(X,x_0) \to \pi_1(X,x_0)/H$ denote the canonical quotient maps. Since I is locally compact Hausdorff, the product map $q_H \times \mathrm{id} : P(X,x_0) \times I \to \widetilde{X}_H \times I$ is a quotient map [36]. Since the graph G_α is assumed to be closed, the set $E_{\alpha,H} = (p_H \times \mathrm{id})^{-1}(G_\alpha)$ is closed in $\widetilde{X}_H \times I$. We have $P_\alpha = (q_H \times \mathrm{id})^{-1}(E_{\alpha,H})$. Therefore, the restriction $(q_H \times \mathrm{id})|_{P_\alpha} : P_\alpha \to E_{\alpha,H}$ is a quotient map. Since $\psi_{\alpha,H} \circ (q_H \times \mathrm{id})|_{P_\alpha} = \pi_H \circ \phi_\alpha$ is continuous and $(q_H \times \mathrm{id})|_{P_\alpha}$ is quotient, $\psi_{\alpha,H}$ is continuous.

$$\begin{array}{ccccc}
P(X,x_0) \times I & \hookleftarrow & P_\alpha & \xrightarrow{\phi_\alpha} & \Omega(X,x_0) \\
\downarrow{\scriptstyle q_H \times \mathrm{id}} & & \downarrow{\scriptstyle (q_H \times \mathrm{id})|_{P_\alpha}} & & \downarrow{\scriptstyle \pi_H} \\
\widetilde{X}_H \times I & \hookleftarrow & E_{\alpha,H} & \xrightarrow{\psi_{\alpha,H}} & \pi_1(X,x_0)/H \\
\downarrow{\scriptstyle p_H \times \mathrm{id}} & & \downarrow & & \\
X \times I & \hookleftarrow & G_\alpha & &
\end{array}$$

■

THEOREM 6.11. *If the graph G_α of every path $\alpha \in P(X,x_0)$ is closed in $X \times I$ (e.g. if X is Hausdorff) and $\pi_1(X,x_0)/H$ is totally path-disconnected, then $p_H : \widetilde{X}_H \to X$ has the continuous path-covering property and $(p_H)_\#(\pi_1(\widetilde{X}_H, \tilde{x}_H)) = H$.*

Proof. Suppose X satisfies the hypotheses of the theorem. Recall that $P(p_H) : P(\widetilde{X}_H, \tilde{x}_H) \to P(X,x_0)$ is a topological retraction by Lemma 6.7. Therefore, it is enough to show that $P(p_H)$ is injective, i.e. $p_H : \widetilde{X}_H \to X$ has the unique path-lifting property. Fix a path $\alpha \in P(X,x_0)$. It suffices to show that the standard lift $\widetilde{\alpha}_H \in P(\widetilde{X}_H, \tilde{x}_H)$ given by $\widetilde{\alpha}_H(t) = H[\alpha_t]$ is the *only* lift of α starting at $\tilde{x}_H$.

Suppose, to obtain a contradiction, that $\widetilde{\beta} : I \to \widetilde{X}_H$ is a lift of α starting at $\tilde{x}_H$ such that $\widetilde{\beta} \neq \widetilde{\alpha}_H$. By restricting the domain if necessary, we may assume that $\widetilde{\beta}(1) \neq \widetilde{\alpha}_H(1)$. Write $\widetilde{\beta}(t) = H[\beta_t]$ for paths $\beta_t : (I,0,1) \to (X,x_0,\alpha(t))$. From the definition of the standard lift, $\widetilde{\beta}(1) \neq \widetilde{\alpha}_H(1)$ implies that $[\beta_1 \cdot \alpha^-] \notin H$. Since $p_H \circ \widetilde{\beta}(t) = \beta_t(1) = \alpha(t)$ for all $t \in I$, there is a well-defined path $\widetilde{\beta}' : I \to E_{\alpha,H}$ given by $\widetilde{\beta}'(t) = (\widetilde{\beta}(t), t)$. By Lemma 6.10, $\psi_{\alpha,H} : E_{\alpha,H} \to \pi_1(X,x_0)/H$ is continuous. Therefore, $\psi_{\alpha,H} \circ \widetilde{\beta}' : I \to \pi_1(X,x_0)/H$ is a continuous path from $\psi_{\alpha,H} \circ \widetilde{\beta}'(0) = \psi_{\alpha,H}(\tilde{x}_H, 0) = H[c_{x_0} \cdot \alpha_0^-] = \tilde{x}_H$ to $\psi_{\alpha,H} \circ \widetilde{\beta}'(1) = \psi_{\alpha,H}(H[\beta_1], 1) = H[\beta_1 \cdot \alpha^-]$. However, $H[\beta_1 \cdot \alpha^-] \neq \tilde{x}_H$, showing that $\psi_{\alpha,H} \circ \widetilde{\beta}'$ is a non-constant path in $\pi_1(X,x_0)/H$, a contradiction of the assumption that $\pi_1(X,x_0)/H$ is totally path-disconnected. We conclude that p_H has the continuous path-covering property.

Finally, note that if $\alpha \in \Omega(X,x_0)$, then the standard lift $\widetilde{\alpha}_H$ is a loop $\Leftrightarrow$ $\widetilde{\alpha}_H(1) = \tilde{x}_H \Leftrightarrow H[\alpha] = H[c_{x_0}] \Leftrightarrow [\alpha] \in H$. It follows from (2) of Lemma 2.4 that $(p_H)_\#(\pi_1(\widetilde{X}_H, \tilde{x}_H)) = H$. ■

Proof of Theorem 1.2. We note that the uniqueness (up to weak equivalence) conditions in Theorem 1.2 are guaranteed by Lemma 6.6. If there exists a map $p : (E, e) \to (X, p(e))$ with the continuous path-covering property such that $H = p_{\#}(\pi_1(E, e))$, then $\pi_1(X, x_0)/H$ is totally path-disconnected by Corollary 5.6. Conversely, if $\pi_1(X, x_0)/H$ is totally path-disconnected, then $p_H : \widetilde{X}_H \to X$ has the continuous path-covering property and satisfies $(p_H)_{\#}(\pi_1(\widetilde{X}_H, \tilde{x}_H)) = H$ by Theorem 6.11. This proves the main statement of Theorem 1.2.

Part (1) follows by combining the main statement with Corollary 3.13. For part (2), recall that $\mathrm{ev}_1 : P(\widetilde{X}_H, \tilde{x}_H) \to \widetilde{X}_H$ is quotient by Lemma 6.7(1). Since every weak equivalence class of maps $E \to X$ with the continuous path-covering property is represented by a map of the form p_H, part (2) follows. ∎

The existence portion of Theorem 1.2 indicates that maps $p : E \to X$ with the continuous path-covering property where E is path-connected exist very often. The next corollary is the case $H = 1$ of Theorem 1.2.

Corollary 6.12. *If X is Hausdorff, then there exists a simply connected space E and a map $p : E \to X$ with the continuous path-covering property if and only if $\pi_1(X, x_0)$ is totally path-disconnected.*

Example 6.13. Consider the canonical homomorphism $\Psi : \pi_1(X, x_0) \to \check{\pi}_1(X, x_0)$ to the first shape homotopy group (see [24, Section 3]). Observe that $\check{\pi}_1(X, x_0)$ is naturally an inverse limit of discrete groups, and with the inverse limit topology on $\check{\pi}_1(X, x_0)$, the natural map Ψ is continuous [8, p. 79]. If Ψ is injective, then $\pi_1(X, x_0)$ continuously injects into an inverse limit of discrete spaces and is therefore totally path-disconnected. By Corollary 6.12, X must admit a simply connected space E and map $p : E \to X$ with the continuous path-covering property. Spaces for which Ψ is injective include, but are not limited to, all one-dimensional spaces [13, 17], planar sets [23], and certain trees of manifolds [22].

7. A remark on topological structure. As noted in the introduction, it is not possible to characterize fibrations with unique path lifting or other maps defined abstractly in terms of unique lifting properties *up to homeomorphism* using the (topologized) fundamental group. However, there are many situations where one can choose a highly structured representative map $p : E \to X$ from a given weak equivalence class.

In this section, we focus on the locally path-connected case. To simplify our terminology, we say that a based map $p : (E, e_0) \to (X, x_0)$ with the continuous path-covering property *corresponds* to a subgroup $H \leq \pi_1(X, x_0)$ if $p_{\#}(\pi_1(E, e_0)) = H$. In what follows, we will implicitly use the fact from

Theorem 1.2 that maps $p_1 : E_1 \to X$ and $p_2 : E_2 \to X$ with the continuous path-covering property are weakly equivalent if and only if for every $e_1 \in E_1$ there exists $e_2 \in E_2$ such that $p_1(e_1) = p_2(e_2)$ and $(p_1)_\#(\pi_1(E_1, e_1)) = (p_2)_\#(\pi_1(E_2, e_2))$.

Inverse limits of path-connected spaces are not always path-connected. Since we only wish to consider non-empty, path-connected domains, the next definition will simplify the exposition to follow.

Definition 7.1. Fix a class $\mathscr{C}$ of maps with the continuous path-covering property. Suppose J is a directed set, $p_j : E_j \to X$, $j \in J$, are maps in $\mathscr{C}$, and $f_{j,j'} : E_j \to E'_j$ are maps satisfying $p_{j'} \circ f_{j,j'} = p_j$ whenever $j \geq j'$ in J and which also satisfy $f_{j',j''} \circ f_{j,j'} = f_{j,j''}$ whenever $j \geq j' \geq j''$. Suppose E is a non-empty path component of the inverse limit $\varprojlim_j (E_j, f_{j,j'})$ and let $p : E \to X$ be the restriction of $\varprojlim_j p_j : \varprojlim_j (E_j, f_{j,j'}) \to X$. We refer to the map p as a *restriction of an inverse limit of maps of type* $\mathscr{C}$. For example, the term *restriction of an inverse limit of covering projections* will refer to maps of the form p where each p_j is a covering projection.

It is straightforward to see that the class of maps with the continuous path-covering property is closed under inverse limits in the above sense. Moreover, the following lemma requires a direct argument involving the universal property of inverse limits; see [7, proof of Lemma 2.31] for details.

Lemma 7.2. *Suppose J is a directed set, $p_j : E_j \to X$ is a map with the continuous path-covering property for every $j \in J$, and $f_{j,j'} : E_j \to E'_j$ are maps satisfying $p_{j'} \circ f_{j,j'} = p_j$ whenever $j \geq j'$ and $f_{j',j''} \circ f_{j,j'} = f_{j,j''}$ whenever $j \geq j' \geq j''$. Let $(e_j)_{j \in J} \in \varprojlim_j (E_j, f_{j,j'})$, E be the path component of $(e_j)_{j\in J}$, and $p : E \to X$ be the restriction of $\varprojlim_j p_j : \varprojlim_j E_j \to X$. Then p has the continuous path-covering property and corresponds to the subgroup $p_\#(\pi_1(E, (e_j))) = \bigcap_{j\in J} (p_j)_\#(\pi_1(E_j, e_j))$.*

Theorem 7.3. *If X is locally path-connected, then a subgroup $H \leq \pi_1(X, x_0)$ is the intersection of open normal subgroups if and only if there exists a map $p : (E, e_0) \to (X, x_0)$ which is a restriction of an inverse limit of regular covering projections and which corresponds to H.*

Proof. We recall [25, Corollary 5.9], which states that a subgroup $K \leq \pi_1(X, x_0)$ contains an open normal subgroup if and only if there exists a covering projection over X that corresponds to K. As a consequence, if $q : (E, e) \to (X, x)$ is a covering projection, then $q_\#(\pi_1(E, e))$ is open in $\pi_1(X, x)$.

Suppose $H = \bigcap_{j\in J} N_j$ where N_j is an open normal subgroup of $\pi_1(X, x_0)$. We may assume $\{N_j \mid j \in J\}$ is the set of all open normal subgroups in

$\pi_1(X,x_0)$ containing H so that J becomes a directed set: $j \geq j'$ in J if and only if $N_j \leq N_{j'}$. For each $j \in J$, find a regular covering projection $p_j : (E_j, e_j) \to (X, x_0)$ corresponding to N_j. Since the spaces E_j are locally path-connected, the usual lifting properties of covering projections give the existence of unique maps $f_{j,j'} : (E_j, e_j) \to (E_{j'}, e_{j'})$ such that $p_{j'} \circ f_{j,j'} = p_j$ whenever $j \geq j'$. Uniqueness of lifts ensures that these maps also satisfy $f_{j',j''} \circ f_{j,j'} = f_{j,j''}$ whenever $j \geq j' \geq j''$. Together, these maps form an inverse system, and taking the limit gives a map $\varprojlim_j p_j : \varprojlim_j E_j \to X$. Let E be the path component of $(e_j)_{j\in J}$ in $\varprojlim_j E_j$. Consider the restriction $f_j : E \to E_j$ of the projection map, and the restriction $p : (E, (e_j)) \to (X, x_0)$ of $\varprojlim_j p_j$. By Lemma 7.2, p has the continuous path-covering property and $p_\#(\pi_1(E,(e_j))) = \bigcap_{j\in J}(p_j)_\#(\pi_1(E_j, e_j)) = \bigcap_{j\in J} N_j = H$.

For the converse, suppose $p : E \to X$ is a restriction of an inverse limit of regular covering projections $p_j : E_j \to X$ where $p : (E, e) \to (X, x)$ corresponds to H. Let $f_j : E \to E_j$ be the projections. Set $e_j = f_j(e)$ and $N_j = (p_j)_\#(\pi_1(E_j, e_j))$. Since p_j is a regular covering projection, N_j is an open normal subgroup of $\pi_1(X, x_0)$ for all $j \in J$. It follows from Lemma 7.2 that $H = \bigcap_{j\in J} N_j$. ■

Definition 7.4 ([5]). A *semicovering map* is a local homeomorphism $p : E \to X$ with the continuous path-covering property.

We refrain from calling a semicovering a "projection" since a semicovering need not be locally trivial. As observed in [34], one may define a semicovering to be a local homeomorphism with the path-covering property (the continuous path-covering property follows as a consequence). Every covering projection over a space X is a semicovering. The converse rarely holds. It even fails for the Infinite Earring used in Example 4.5; see [25]. We prove the following theorem, which generalizes the classical theorem that every covering projection is a Hurewicz fibration [35, Theorem 2.2.3]; the lack of local triviality requires a line of argument different from Spanier's proof.

Theorem 7.5. *Every semicovering map is a Hurewicz fibration with discrete fibers.*

Proof. Let $p : E \to X$ be a semicovering map. Since p is a local homeomorphism, p has discrete fibers. It suffices to check that p has the homotopy lifting property with respect to an arbitrary space Z. Let $f : Z \to E$ and $g : Z \times I \to X$ be maps such that $p(f(z)) = g(z, 0)$. For each $z \in Z$, let $\gamma_z : I \to X$ denote the path given by $g(z,t)$ and let $\widetilde{\gamma}_z : I \to E$ be the unique continuous lift such that $\widetilde{\gamma}_z(0) = f(z)$. This gives a function $\widetilde{g} : Z \times I \to E$ defined by $\widetilde{g}(z,t) = \widetilde{\gamma}_z(t)$. Since $p \circ \widetilde{g} = g$, it suffices to show that $\widetilde{g}$ is continuous. We do this by showing that $\widetilde{g}$ is continuous on each member of an open cover of $Z \times I$ by sets of the form $V \times I$.

Fix $z_0 \in Z$. Since the path $\widetilde{\gamma}_{z_0} : I \to E$ is continuous, we may find a subdivision $0 = t_0 < t_1 < \cdots < t_m = 1$ and open sets $U_1, \ldots, U_m$ such that $\widetilde{\gamma}_{z_0}([t_{j-1}, t_j]) \subseteq U_j$ and that p maps U_j homeomorphically onto the open set $p(U_j)$ of X. Find an open neighborhood A_1 of z_0 in Z such that $f(A_1) \subseteq U_1$. Since $\gamma_{z_0}([t_0, t_1]) \subseteq p(U_1)$, the compactness of $[t_0, t_1]$ and the continuity of g allow us to find a neighborhood B_1 of z_0 in Z such that $B_1 \subseteq A_1$ and $g(B_1 \times [t_0, t_1]) \subseteq p(U_1)$. Since $\widetilde{g}(B_1 \times \{t_0\}) = f(B_1) \subseteq U_1$, we have $\widetilde{g}|_{V_1' \times [t_0,t_1]} = p|_{U_j}^{-1} \circ g|_{B_1 \times [t_0,t_1]}$, so we may conclude that $\widetilde{g}$ is continuous on $B_1 \times [t_0, t_1]$. Since $g(z_0, t_1) = \gamma_{z_0}(t_1) \in p(U_1 \cap U_2)$, we may find a neighborhood A_2 of z_0 in Z such that $A_2 \subseteq B_1$ and $g(A_2 \times \{t_1\}) \subseteq p(U_1 \cap U_2)$. Since p maps $U_1 \cap U_2$ homeomorphically onto the open set $p(U_1 \cap U_2)$ (but not necessarily onto $p(U_1) \cap p(U_2)$), our choice ensures that $\widetilde{g}$ is continuous on $A_2 \times [t_0, t_1]$ and $\widetilde{g}(A_2 \times \{t_1\}) \subseteq U_1 \cap U_2$.

Applying the same procedure, we may find neighborhoods $z_0 \in A_3 \subseteq B_2 \subseteq A_2$ such that $\widetilde{g}$ is continuous on $A_3 \times [t_1, t_2]$ and $\widetilde{g}(A_3 \times \{t_2\}) \subseteq U_2 \cap U_3$. Proceeding inductively, we obtain finitely many nested neighborhoods $z_0 \in A_n \subseteq A_{n-1} \subseteq \cdots \subseteq A_2 \subseteq A_1$ such that $\widetilde{g}$ is continuous on $A_j \times [t_{j-2}, t_{j-1}]$ and $\widetilde{g}(A_j \times \{t_{j-1}\}) \subseteq U_{j-1} \cap U_j$ (the second inclusion being required for the induction). We terminate the induction with A_{n+1} by taking $U_{n+1} = U_n$.

Let $V = A_{n+1}$. By restricting $\widetilde{g}$, we see that $\widetilde{g}$ is continuous on $V \times [t_{j-1}, t_j]$ for all $j \in \{1, \ldots, n\}$. Hence, by the pasting lemma, $\widetilde{g}$ is continuous on the tube $V \times I$. Since $\widetilde{g}$ is continuous on an open neighborhood of every point in $Z \times I$, $\widetilde{g}$ is continuous. ∎

Since the class of Hurewicz fibrations with the unique path-lifting property is closed under forming restrictions of inverse limits (in the sense of Definition 7.1), we have the following corollary.

Corollary 7.6. *If $p : E \to X$ is a restriction of an inverse limit of semicovering maps, then p is a Hurewicz fibration.*

Remark 7.7. If $p : (E, e_0) \to (X, x_0)$ is an arbitrary semicovering, then $p_\#(\pi_1(E, e_0))$ is an open subgroup of $\pi_1(X, x_0)$ [5, proof of Theorem 5.5]. For a locally path-connected space X (and many non-locally-path-connected spaces), the open subgroups of $\pi_1(X, x_0)$ are classified by the semicovering maps over X (see [5] again). By mimicking the proof of Theorem 7.3, one may prove an analogous statement for inverse limits: H is the intersection of open (not necessarily normal) subgroups if and only if there exists an inverse limit $p : (E, e_0) \to (X, x_0)$ of semicovering maps that corresponds to H.

Definition 7.8 ([4]). The *τ-topology* on $\pi_1(X, x_0)$ is the finest topology such that (1) $\pi_1(X, x_0)$ is a topological group, and (2) the map $q : \Omega(X, x_0) \to \pi_1(X, x_0)$, $q(\alpha) = [\alpha]$ is continuous. We write $\pi_1^\tau(X, x_0)$ for the fundamental group equipped with the τ-topology.

Note that the quotient topology of $\pi_1(X,x_0)$ is generally finer than that of $\pi_1^\tau(X,x_0)$ and we have $\pi_1(X,x_0)=\pi_1^\tau(X,x_0)$ if and only if $\pi_1(X,x_0)$ is a topological group. It is shown in [4, Corollary 3.9] that the groups $\pi_1(X,x_0)$ and $\pi_1^\tau(X,x_0)$ share the same open subgroups. Additionally, every topological group G is isomorphic to $\pi_1^\tau(X,x_0)$ for some space X constructed in a manner similar to a 2-dimensional CW-complex, that is, by attaching 2-cells to a generalized wedge of circles [4, Example 4.16]. In particular, such spaces X satisfy a property called *wep-connectedness*, which is introduced in [5], and which lies between X being locally path-connected and $\mathrm{ev}_1 : P(X,x_0)\to X$ being quotient [5, Prop. 6.2]. We will only need to use the following weaker fact: every topological group G is isomorphic to $\pi_1^\tau(X,x_0)$ for a space X where $\mathrm{ev}_1 : P(X,x_0)\to X$ is a quotient map.

Example 7.9. We give an extreme example to show that there are maps with the continuous path-covering property that cannot be restrictions of inverse limits of covering projections. Find a space X such that $\mathrm{ev}_1 : P(X,x_0)\to X$ is quotient and $\pi_1^\tau(X,x_0)$ is topologically isomorphic to the additive group of rationals $\mathbb{Q}$. Since the quotient topology of $\pi_1(X,x_0)$ is finer than the τ-topology, $\pi_1(X,x_0)$ is totally path-disconnected. Moreover, if $H\leq\pi_1(X,x_0)$ is closed, then $\pi_1(X,x_0)/H$ is a countable T_1 space and therefore must be totally path-disconnected. Therefore, the closed subgroups of $\mathbb{Q}$ are classified by maps $E\to X$ with the continuous path-covering property up to weak equivalence (and up to equivalence if we restrict to total spaces E with $\mathrm{ev}_1 : P(E,e)\to E$ quotient). However, $\mathbb{Q}$ has no proper open subgroups. Since $\pi_1(X,x_0)$ and $\pi_1^\tau(X,x_0)$ share the same open subgroups, $\pi_1(X,x_0)$ has no proper open subgroups. Therefore, X admits many maps $E\to X$ with the continuous path-covering property, but Remark 7.7 ensures that the identity map $X\to X$ is the only one which is equivalent to a (semi)covering or an inverse limit of (semi)coverings.

We consider restrictions of inverse limits of (semi)coverings to be “highly structured” among those maps with the continuous path-covering property. To prove Theorem 1.3, we apply some famous structure theorems from topological group theory.

Proof of Theorem 1.3. We only consider the non-trivial case where X is not simply connected. Since p has the continuous path-covering property, $\pi_1(X,x_0)/H$ is a totally path-disconnected, T_1 quasitopological group (recall the proofs of Proposition 3.3 and Corollary 5.6). If $\pi_1(X,x_0)/H$ is compact or, more generally, locally compact, then the quasitopological group $\pi_1(X,x_0)/H$ is a topological group by a theorem of R. Ellis [18]. Hence, in both statements (1) and (2) to be proven, $\pi_1(X,x_0)/H$ is a non-trivial, locally compact, totally path-disconnected, Hausdorff topological group. It is a result of Gleason [28] that every locally compact group that is not totally disconnected must contain an arc. Hence, $\pi_1(X, x_0)/H$ must also be totally disconnected.

(1) It is well-known that every totally disconnected compact group is a pro-finite group [32, Theorem 1.34]. Therefore, if $\pi_1(X, x_0)/H$ is compact, it is pro-finite, and the identity element of $\pi_1(X, x_0)/H$ has a basis $\{N_j \mid j \in J\}$ of finite-index open normal subgroups (whose intersection is the trivial subgroup). If $q : \pi_1(X, x_0) \to \pi_1(X, x_0)/H$ is the natural quotient map, then it follows that H is equal to the intersection $\bigcap_{j\in J} q^{-1}(N_j)$ of open subgroups $q^{-1}(N_j)$, which are finite-index and normal in $\pi_1(X, x_0)$. Since X is locally path-connected, by [25, Corollary 5.9], there exist finite-sheeted regular covering maps $p_j : E_j \to X$ corresponding to the subgroups $q^{-1}(N_j)$ respectively. Applying Theorem 7.3, we see that p is weakly equivalent to the restriction of an inverse limit of finite-sheeted regular covering projections as defined in Definition 7.1.

(2) If E is simply connected, or equivalently if $H = 1$, and $\pi_1(X, x_0) = \pi_1(X, x_0)/H$ is locally compact, then by van Dantzig's Theorem [15], there exists a compact open subgroup $K \leq \pi_1(X, x_0)$. Since X is locally path-connected, the classification of semicoverings [5] applies and there exists a semicovering $q : (E', e'_0) \to (X, x_0)$ such that $K = q_\#(\pi_1(E', e'_0))$. Note that E' is locally path-connected since q is a local homeomorphism. By statement (2) of Theorem 1.2, we may assume that $\mathrm{ev}_1 : P(E, e_0) \to E$ is quotient without changing the weak equivalence class of p. Since E is simply connected, Lemma 3.12 gives a unique map $r : (E, e_0) \to (E', e'_0)$ such that $q \circ r = p$. Since p and q have the continuous path-covering property, it follows from Lemma 3.5 that r also has the continuous path-covering property. By the proof of Theorem 5.2, $q_\#$ maps $\pi_1(E', e'_0)$ homeomorphically onto K. In particular, $\pi_1(E', e'_0)$ is compact. By part (1), r is weakly equivalent to the restriction of an inverse limit of covering projections $r_j : (E_j, e_j) \to (E', e'_0)$, $j \in J$, over E'. The composition of two semicoverings is a semicovering [5, Cor. 3.5] (but need not be a covering projection), and thus $q \circ r_j : E_j \to X$ is a semicovering for all $j \in J$. Since $\bigcap_{j\in J} (r_j)_\#(\pi_1(E_j, e_j)) = 1$ and $q_\#$ is injective, we have

$$\bigcap_{j\in J} (q \circ r_j)_\#(\pi_1(E_j, e_j)) = q_\#\Big(\bigcap_{j\in J} (r_j)_\#(\pi_1(E_j, e_j))\Big) = 1.$$

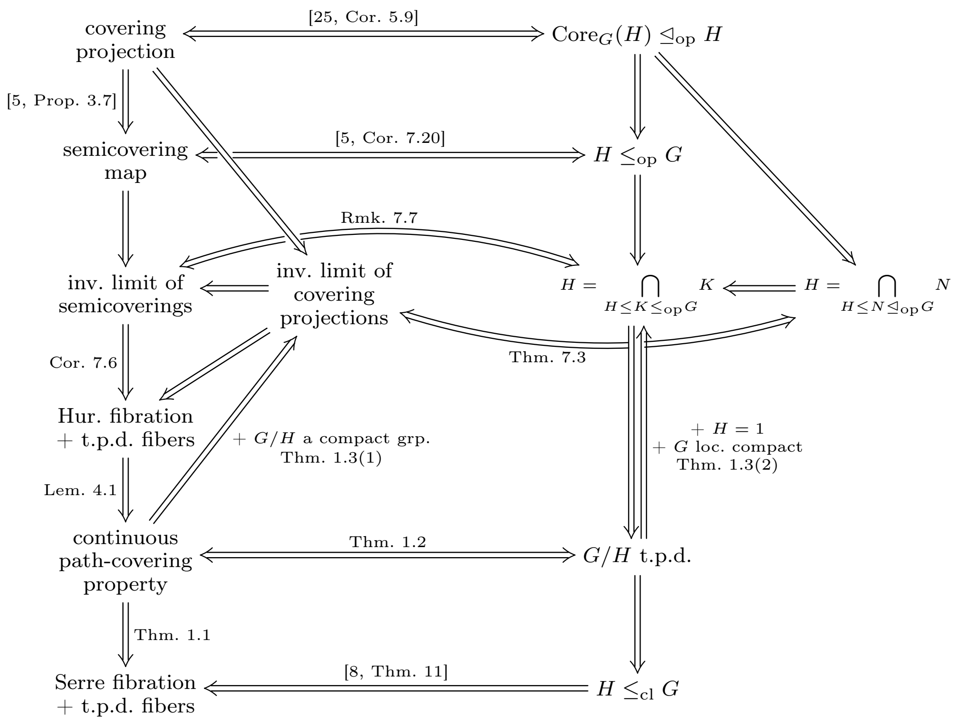
Applying Lemma 7.2 and Theorem 1.2, we deduce that p is weakly equivalent to the restriction of an inverse limit of the semicovering maps $q \circ r_j$. ∎

8. Diagrammatic summary. We conclude with a diagram that summarizes the relationships between lifting properties studied in this paper and the topology of $G = \pi_1(X, x_0)$. In the diagram, X is assumed to be locally path-connected, $p : (E, e_0) \to (X, x_0)$ is a map and $H = p_\#(\pi_1(E, e_0))$. The left side of the diagram involves properties of p and the right side of the diagram involves properties of H. We give the following key for reading the diagram:

- $A \leq_{\mathrm{cl}} B$: A is a closed subgroup of B,
- $A \leq_{\mathrm{op}} B$: A is an open subgroup of B,
- $A \trianglelefteq_{\mathrm{op}} B$: A is an open normal subgroup of B,
- t.p.d.: totally path-disconnected,
- $\mathrm{Core}_G(H) = \bigcap_{g\in G} gHg^{-1}$: the core of H in G, i.e. the largest normal subgroup of G that is a subgroup of H.

A horizontal biconditional arrow means that there exists a map weakly equivalent to p that satisfies the property on the left if and only if H satisfies the property on the right. For example, p is weakly equivalent to a covering projection if and only if the core of H in G is open, which is equivalent to the shorter statement $\mathrm{Core}_G(H) \trianglelefteq_{\mathrm{op}} H$. Notably, all horizontal arrows are biconditional except for the last one.

All horizontal or downward arrows hold without extra hypotheses. The two partial converse arrows that point upward (from Theorem 1.3) are labeled with the extra required hypotheses (indicated with a "+") and appear on opposite sides of the diagram to minimize clutter in the image.



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