2012

A Reciprocity Relation for WP-Bailey Pairs

James McLaughlin

Peter Zimmer

A RECIPROCITY RELATION FOR WP-BAILEY PAIRS

JAMES MC LAUGHLIN AND PETER ZIMMER

Abstract. We derive a new general transformation for WP-Bailey pairs by considering the a certain limiting case of a WP-Bailey chain previously found by the authors, and examine several consequences of this new transformation. These consequences include new summation formulae involving WP-Bailey pairs.

Other consequences include new proofs of some classical identities due to Jacobi, Ramanujan and others, and indeed extend these identities to identities involving particular specializations of arbitrary WP-Bailey pairs.

1. Introduction

In the present paper, we derive a new general transformation for WP-Bailey pairs by considering the a certain limiting case of a WP-Bailey chain previously found by the authors, and examine several consequences of this new transformation. These consequences include new expressions for various theta functions and new summation formulae involving WP-Bailey pairs.

A *WP-Bailey pair* (see Andrews [1]) is a pair of sequences $(\alpha_n(a,k,q)$, $\beta_n(a,k,q))$ satisfying $\alpha_0(a,k,q)=\beta_0(a,k,q)=1$, and for $n>0$,

$$\beta_n(a,k,q)=\sum_{j=0}^{n}\frac{(k/a;q)_{n-j}(k;q)_{n+j}}{(q;q)_{n-j}(aq;q)_{n+j}}\alpha_j(a,k,q). \tag{1.1}$$

If the context is clear, we occasionally suppress the dependence on some or all of a, k and q. For a WP-Bailey pair $(\alpha_n(a,k),\beta_n(a,k))$, define

(1.2)

$$F(a,k,q):=\sum_{n=1}^{\infty}\frac{(1-kq^{2n})(q;q)_{n-1}\left(\frac{k^2}{a};q\right)_n(qa;q^2)_n}{(1-k)\left(\frac{qa}{k},kq;q\right)_n\left(\frac{k^2q}{a};q^2\right)_n}\left(\frac{-qa}{k}\right)^n\beta_n(a,k)$$
$$-\sum_{n=1}^{\infty}\frac{(q^2;q^2)_{n-1}\left(\frac{k^2}{a};q^2\right)_n}{\left(\frac{q^2a^2}{k^2},q^2a;q^2\right)_n}\left(\frac{qa}{k}\right)^{2n}\alpha_{2n}(a,k)$$

Date: April 15, 2011.

2000 *Mathematics Subject Classification.* Primary: 33D15. Secondary:11B65, 05A19.

Key words and phrases. Bailey pairs, WP-Bailey Chains, WP-Bailey pairs, Lambert Series, Basic Hypergeometric Series, q-series, theta series, q-products.

$$+\frac{\left(\frac{k^2}{a},\frac{q^3a^2}{k^2},q^3a,q^2;q^2\right)_\infty}{\left(\frac{k^2q}{a},\frac{q^2a^2}{k^2},q^2a,q;q^2\right)_\infty}\sum_{n=0}^{\infty}\frac{\left(\frac{k^2q}{a},q;q^2\right)_n}{\left(\frac{q^3a^2}{k^2},q^3a;q^2\right)_n}\left(\frac{qa}{k}\right)^{2n+1}\alpha_{2n+1}(a,k).$$

The main result of the paper is the following reciprocity result for the function $F(a,k,q)$.

Theorem 1. *Let a and k be non-zero complex numbers and $|q|$ a complex number such that $|q|<\max\{1,|a/k|,|k/a|\}$ and none of the denominators following vanish. Let $(\alpha_n(a,k),\beta_n(a,k))$ be a WP-Bailey pair. Let $F(a,k,q)$ be as at* (1.2). *Then*

$$\begin{aligned}(1.3)\quad F(a,k,q)-F\left(\frac{1}{a},\frac{1}{k},q\right)&\\ =\frac{aq}{k^2}\frac{(aq,q/a,k^2/a,q^2a/k^2,k^2/q,q^3/k^2,q^2,q^2;q^2)_\infty}{(aq/k^2,k^2q/a,a,q^2/a,k^2,q^2/k^2,q,q;q^2)_\infty}&\\ -\frac{a}{k}\frac{(k^2/a,qa/k^2,-a,-q/a;q)_\infty(q^2,q^2;q^2)_\infty}{(k^2,q^2/k^2,a^2/k^2,q^2k^2/a^2;q^2)_\infty}+\frac{(a^2-k)(a-k^2)}{(1-a)(1-k)(a^2-k^2)}.&\end{aligned}$$

Implications of this result include new representations for various theta functions. One example is contained in the following identity. Recall that

$$(1.4)\qquad \psi(q)=\sum_{n=0}^{\infty}q^{n(n+1)/2}=\frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}$$

is Ramanujan's theta function. Let $\omega:=\exp(2\pi i/3)$, let $\chi_0(n)$ denote the principal character modulo 3, and let $\psi(q)$ be as defined at (1.4). Then

$$\begin{aligned}9\sum_{n=1}^{\infty}\chi_0(n)\frac{nq^n}{1-q^{2n}}&=\frac{\psi^6(q)}{\psi^2(q^3)}-\frac{\psi^3(q^{1/2})\psi^3(-q^{1/2})}{\psi(q^{3/2})\psi(-q^{3/2})}\\ &=3(1-\omega^2)\sum_{n=1}^{\infty}\frac{(1-\omega q^{2n})(q;q)_{n-1}(\omega^2q;q^2)_n(-\omega q)^n}{(1-q^{3n})(\omega q;q)_n(q;q^2)_n}\\ &\quad+3(1-\omega)\sum_{n=1}^{\infty}\frac{(1-\omega^2q^{2n})(q;q)_{n-1}(\omega q;q^2)_n(-\omega^2q)^n}{(1-q^{3n})(\omega^2q;q)_n(q;q^2)_n}.\end{aligned}$$

Another implication is the following transformation for WP-Bailey pairs. If $(\alpha_n(a,k,q),\beta_n(a,k,q))$ is a WP-Bailey pair, then

$$\begin{aligned}&\sum_{n=1}^{\infty}\frac{(1-kq^{2n})(q;q)_{n-1}\left(\frac{k^2}{q};q\right)_n(q^2;q^2)_n}{(1-k)\left(\frac{q^2}{k},kq;q\right)_n(k^2;q^2)_n}\left(\frac{-q^2}{k}\right)^n\beta_n(q,k,q)\\ &\quad-\sum_{n=1}^{\infty}\frac{(q^2;q^2)_{n-1}\left(\frac{k^2}{q};q^2\right)_n q^{4n}\alpha_{2n}(q,k,q)}{\left(\frac{q^4}{k^2},q^3;q^2\right)_n k^{2n}}+\frac{\left(k^2-q\right)\left(k-q^2\right)}{(1-k)(1-q)(k^2-q^2)}\end{aligned}$$

$$= k\frac{(k^2q, q/k^2, q^2, q^2; q^2)_\infty}{(k^2, q^2/k^2, q, q; q^2)_\infty}\left(1+\sum_{n=0}^{\infty}\frac{\left(\frac{k^2}{q^2}, \frac{1}{q}; q^2\right)_{n+1} q^{4n+4}\alpha_{2n+1}(q,k,q)}{\left(\frac{q^3}{k^2}, q^2; q^2\right)_{n+1} k^{2n+2}}\right).$$

Several other implications are to be found elsewhere in the paper.

2. BACKGROUND

The subject of basic hypergeometric series took a leap forward after Andrews development in [1] of a *WP-Bailey Chain*, a mechanism for deriving new WP-bailey pairs from existing pairs. In [1], Andrews in fact describes two such chains. Warnaar [15] found four additional chains and Liu and Ma [8] introduced the idea of a *general WP-Bailey chain*, and discovered one new specific WP-Bailey chain. In [13], the authors found three new WP-Bailey chains.

Each WP-Bailey chain also implies a transformation connecting the terms in an arbitrary WP-Bailey pair (see (3.1) below for an example of such a transformation), leading to new transformation formulae for basic hypergeometric series.

In [9], the first author of the present paper went in a somewhat different direction and found two new types of transformations for WP-Bailey pairs.

Theorem 2 (McL., [9]). *If $(\alpha_n(a,k), \beta_n(a,k))$ is a WP-Bailey pair, then subject to suitable convergence conditions,*

$$\sum_{n=1}^{\infty}\frac{(q\sqrt{k}, -q\sqrt{k}, z; q)_n (q;q)_{n-1}}{\left(\sqrt{k}, -\sqrt{k}, qk, \frac{qk}{z}; q\right)_n}\left(\frac{qa}{z}\right)^n\beta_n(a,k) \tag{2.1}$$
$$-\sum_{n=1}^{\infty}\frac{\left(q\sqrt{\frac{1}{k}}, -q\sqrt{\frac{1}{k}}, \frac{1}{z}; q\right)_n (q;q)_{n-1}}{\left(\sqrt{\frac{1}{k}}, -\sqrt{\frac{1}{k}}, \frac{q}{k}, \frac{qz}{k}; q\right)_n}\left(\frac{qz}{a}\right)^n\beta_n\left(\frac{1}{a},\frac{1}{k}\right)-$$
$$\sum_{n=1}^{\infty}\frac{(z;q)_n(q;q)_{n-1}}{\left(qa, \frac{qa}{z}; q\right)_n}\left(\frac{qa}{z}\right)^n\alpha_n(a,k)+\sum_{n=1}^{\infty}\frac{\left(\frac{1}{z};q\right)_n(q;q)_{n-1}}{\left(\frac{q}{a}, \frac{qz}{a}; q\right)_n}\left(\frac{qz}{a}\right)^n\alpha_n\left(\frac{1}{a},\frac{1}{k}\right)$$
$$=\frac{(a-k)\left(1-\frac{1}{z}\right)\left(1-\frac{ak}{z}\right)}{(1-a)(1-k)\left(1-\frac{a}{z}\right)\left(1-\frac{k}{z}\right)}+\frac{z}{k}\frac{\left(z, \frac{q}{z}, \frac{k}{a}, \frac{qa}{k}, \frac{ak}{z}, \frac{qz}{ak}, q, q; q\right)_\infty}{\left(\frac{z}{k}, \frac{qk}{z}, \frac{z}{a}, \frac{qa}{z}, a, \frac{q}{a}, k, \frac{q}{k}; q\right)_\infty}.$$

Theorem 3 (McL., [9]). *If $(\alpha_n(a,k,q), \beta_n(a,k,q))$ is a WP-Bailey pair, then subject to suitable convergence conditions,*

$$\sum_{n=1}^{\infty}\frac{(1-kq^{2n})(z;q)_n(q;q)_{n-1}}{(1-k)(qk, qk/z; q)_n}\left(\frac{qa}{z}\right)^n\beta_n(a,k,q) \tag{2.2}$$
$$+\sum_{n=1}^{\infty}\frac{(1+kq^{2n})(z;q)_n(q;q)_{n-1}}{(1+k)(-qk, -qk/z; q)_n}\left(\frac{-qa}{z}\right)^n\beta_n(-a,-k,q)$$

$$
\begin{aligned}
&- 2\sum_{n=1}^{\infty}\frac{(1-k^2q^{4n})(z^2;q^2)_n(q^2;q^2)_{n-1}}{(1-k^2)(q^2k^2,q^2k^2/z^2;q^2)_n}\left(\frac{q^2a^2}{z^2}\right)^n\beta_n(a^2,k^2,q^2)\\
&\qquad = \sum_{n=1}^{\infty}\frac{(z;q)_n(q;q)_{n-1}}{(qa,qa/z;q)_n}\left(\frac{qa}{z}\right)^n\alpha_n(a,k,q)\\
&\qquad + \sum_{n=1}^{\infty}\frac{(z;q)_n(q;q)_{n-1}}{(-qa,-qa/z;q)_n}\left(\frac{-qa}{z}\right)^n\alpha_n(-a,-k,q)\\
&\qquad - 2\sum_{n=1}^{\infty}\frac{(z^2;q^2)_n(q^2;q^2)_{n-1}}{(q^2a^2,q^2a^2/z^2;q^2)_n}\left(\frac{q^2a^2}{z^2}\right)^n\alpha_n(a^2,k^2,q^2).
\end{aligned}
$$

Some similar results were obtained in two other papers, [10, 11], and various consequences of the transformations found were also examined. The results in the present paper may be viewed as deriving from a continuation of the investigations in the papers alluded to above.

3. Proof of the main identity

Before coming to the proofs, we recall an identity derived by the present authors in [12].

Theorem 4. ([12, Mc Laughlin, Zimmer]) *If $(\alpha_n(a,k),\beta_n(a,k)$ is a WP-Bailey pair, then*

$$
\begin{aligned}
&\frac{(qab/k,kq/b;q)_\infty}{(kq,qa/k;q)_\infty}\\
&\quad\times\sum_{n=0}^{\infty}\frac{(q\sqrt{k},-q\sqrt{k},k^2/ab,b,\sqrt{qa},-\sqrt{qa};q)_n}{(\sqrt{k},-\sqrt{k},qab/k,kq/b,k\sqrt{q/a},-k\sqrt{q/a};q)_n}\left(\frac{-qa}{k}\right)^n\beta_n\\
&=\frac{\left(\frac{qk^2}{ab},bq,\frac{q^2a^2b}{k^2},\frac{q^2a}{b};q^2\right)_\infty}{\left(q,\frac{k^2q}{a},q^2a,\frac{q^2a^2}{k^2};q^2\right)_\infty}\sum_{n=0}^{\infty}\frac{\left(\frac{k^2}{ab},b;q^2\right)_n}{\left(\frac{q^2a^2b}{k^2},\frac{q^2a}{b};q^2\right)_n}\left(\frac{-qa}{k}\right)^{2n}\alpha_{2n}\\
&\quad+\frac{\left(\frac{k^2}{ab},b,\frac{q^3a^2b}{k^2},\frac{q^3a}{b};q^2\right)_\infty}{\left(q,\frac{k^2q}{a},q^2a,\frac{q^2a^2}{k^2};q^2\right)_\infty}\sum_{n=0}^{\infty}\frac{\left(\frac{k^2q}{ab},bq;q^2\right)_n}{\left(\frac{q^3a^2b}{k^2},\frac{q^3a}{b};q^2\right)_n}\left(\frac{-qa}{k}\right)^{2n+1}\alpha_{2n+1}.
\end{aligned}
\tag{3.1}
$$

We also recall a result from [9], namely that if $f(a,k,z,q)$ is as defined by

$$
\begin{aligned}
f(a,k,z,q) &= \sum_{n=1}^{\infty}\frac{(q\sqrt{k},-q\sqrt{k},k,z,k/a;q)_n}{(\sqrt{k},-\sqrt{k},qk,qk/z,qa;q)_n(1-q^n)}\left(\frac{qa}{z}\right)^n\\
&= -\sum_{n=1}^{\infty}\frac{(q\sqrt{a},-q\sqrt{a},a,z,a/k;q)_n}{(\sqrt{a},-\sqrt{a},qa,qa/z,qk;q)_n(1-q^n)}\left(\frac{qk}{z}\right)^n\\
&= \sum_{n=1}^{\infty}\frac{kq^n}{1-kq^n}+\sum_{n=1}^{\infty}\frac{q^na/z}{1-q^na/z}-\sum_{n=1}^{\infty}\frac{aq^n}{1-aq^n}-\sum_{n=1}^{\infty}\frac{q^nk/z}{1-q^nk/z},
\end{aligned}
\tag{3.2}
$$

then

$$f(a,k,z,q)-f\left(\frac{1}{a},\frac{1}{k},\frac{1}{z},q\right)=\frac{(a-k)(1-1/z)(1-ak/z)}{(1-a)(1-k)(1-a/z)(1-k/z)}+\frac{z}{k}\frac{(z,q/z,k/a,qa/k,ak/z,qz/ak,q,q;q)_\infty}{(z/k,qk/z,z/a,qa/z,a,q/a,k,q/k;q)_\infty}. \tag{3.3}$$

We remark that the identity (3.3) was also proved by the authors in [3], for the final form of $f(a,k,z,q)$ above. We note two special cases for the proof below. Firstly,

$$f(a/k,k,-1,q)=2\sum_{n=1}^{\infty}\frac{kq^n}{1-k^2q^{2n}}-2\sum_{n=1}^{\infty}\frac{aq^n/k}{1-a^2q^{2n}/k^2} \tag{3.4}$$

$$\begin{aligned}&f(a/k,k,-1,q)-f\left(k/a,1/k,-1,q\right)\\&=2\frac{(a/k-k)(1+a)}{(1-a^2/k^2)(1-k^2)}-2\frac{a}{k}\frac{(k^2/a,qa/k^2,-a,-q/a;q)_\infty(q^2,q^2;q^2)_\infty}{(k^2,q^2/k^2,a^2/k^2,q^2k^2/a^2;q^2)_\infty}.\end{aligned} \tag{3.5}$$

Secondly,

$$\begin{aligned}&f\left(a,k^2,aq,q^2\right)\\&=\sum_{n=1}^{\infty}\frac{k^2q^{2n}}{1-k^2q^{2n}}+\sum_{n=1}^{\infty}\frac{q^{2n}/q}{1-q^{2n}/q}-\sum_{n=1}^{\infty}\frac{aq^{2n}}{1-aq^{2n}}-\sum_{n=1}^{\infty}\frac{\frac{k^2}{aq}q^{2n}}{1-\frac{k^2}{aq}q^{2n}},\end{aligned} \tag{3.6}$$

$$\begin{aligned}&f\left(a,k^2,aq,q^2\right)-f\left(\frac{1}{a},\frac{1}{k^2},\frac{1}{aq},q^2\right)\\&=\frac{(a-k^2)\left(1-\frac{1}{aq}\right)\left(1-\frac{k^2}{q}\right)}{(1-a)(1-k^2)\left(1-\frac{1}{q}\right)\left(1-\frac{k^2}{aq}\right)}\\&\quad+\frac{aq}{k^2}\frac{(aq,q/a,k^2/a,q^2a/k^2,k^2/q,q^3/k^2,q^2,q^2;q^2)_\infty}{(aq/k^2,k^2q/a,q,q,a,q^2/a,k^2,q^2/k^2;q^2)_\infty}.\end{aligned} \tag{3.7}$$

Proof of Theorem 1. Rewrite (3.1) as

$$\begin{aligned}&\frac{(qab/k,kq/b;q)_\infty}{(kq,qa/k;q)_\infty}\\&\times\sum_{n=1}^{\infty}\frac{(1-kq^{2n})(k^2/ab,b;q)_n(qa;q^2)_n}{(1-k)(qab/k,kq/b;q)_n(k^2q/a;q^2)_n}\left(\frac{-qa}{k}\right)^n\beta_n(a,k)\\&-\frac{\left(\frac{qk^2}{ab},bq,\frac{q^2a^2b}{k^2},\frac{q^2a}{b};q^2\right)_\infty}{\left(q,\frac{k^2q}{a},q^2a,\frac{q^2a^2}{k^2};q^2\right)_\infty}\sum_{n=1}^{\infty}\frac{\left(\frac{k^2}{ab},b;q^2\right)_n}{\left(\frac{q^2a^2b}{k^2},\frac{q^2a}{b};q^2\right)_n}\left(\frac{qa}{k}\right)^{2n}\alpha_{2n}(a,k)\end{aligned} \tag{3.8}$$

$$+\frac{\left(\frac{k^2}{ab}, b, \frac{q^3a^2b}{k^2}, \frac{q^3a}{b}; q^2\right)_\infty}{\left(q, \frac{k^2q}{a}, q^2a, \frac{q^2a^2}{k^2}; q^2\right)_\infty} \sum_{n=0}^{\infty} \frac{\left(\frac{k^2q}{ab}, bq; q^2\right)_n}{\left(\frac{q^3a^2b}{k^2}, \frac{q^3a}{b}; q^2\right)_n} \left(\frac{qa}{k}\right)^{2n+1} \alpha_{2n+1}(a,k)$$
$$= \frac{\left(\frac{qk^2}{ab}, bq, \frac{q^2a^2b}{k^2}, \frac{q^2a}{b}; q^2\right)_\infty}{\left(q, \frac{k^2q}{a}, q^2a, \frac{q^2a^2}{k^2}; q^2\right)_\infty} - \frac{(qab/k, kq/b; q)_\infty}{(kq, qa/k; q)_\infty}.$$

If we divide through by $1-b$ and then let $b \to 1$ on the left side of (3.8), then the result is $F(a,k,q)$. If we define

$$H(b) := \frac{(qab/k, kq/b; q)_\infty}{(kq, qa/k; q)_\infty} \frac{\left(q, \frac{k^2q}{a}, q^2a, \frac{q^2a^2}{k^2}; q^2\right)_\infty}{\left(\frac{qk^2}{ab}, bq, \frac{q^2a^2b}{k^2}, \frac{q^2a}{b}; q^2\right)_\infty}$$

then we see that dividing the right side of (3.8) by $1-b$ gives

$$\frac{\left(\frac{qk^2}{ab}, bq, \frac{q^2a^2b}{k^2}, \frac{q^2a}{b}; q^2\right)_\infty}{\left(q, \frac{k^2q}{a}, q^2a, \frac{q^2a^2}{k^2}; q^2\right)_\infty} \frac{1-H(b)}{1-b},$$

so that the result of letting $b \to 1$ is

(3.9)

$$\begin{aligned}
H'(1) &= \sum_{n=1}^{\infty} \frac{kq^n}{1-kq^n} - \sum_{n=1}^{\infty} \frac{aq^n/k}{1-aq^n/k} \\
&\quad + \sum_{n=1}^{\infty} \frac{a^2q^{2n}/k^2}{1-a^2q^{2n}/k^2} + \sum_{n=1}^{\infty} \frac{q^{2n}/q}{1-q^{2n}/q} - \sum_{n=1}^{\infty} \frac{aq^{2n}}{1-aq^{2n}} - \sum_{n=1}^{\infty} \frac{\frac{k^2}{aq}q^{2n}}{1-\frac{k^2}{aq}q^{2n}} \\
&= \sum_{n=1}^{\infty} \frac{kq^n}{1-k^2q^{2n}} - \sum_{n=1}^{\infty} \frac{aq^n/k}{1-a^2q^{2n}/k^2} \\
&\quad + \sum_{n=1}^{\infty} \frac{k^2q^{2n}}{1-k^2q^{2n}} + \sum_{n=1}^{\infty} \frac{q^{2n}/q}{1-q^{2n}/q} - \sum_{n=1}^{\infty} \frac{aq^{2n}}{1-aq^{2n}} - \sum_{n=1}^{\infty} \frac{\frac{k^2}{aq}q^{2n}}{1-\frac{k^2}{aq}q^{2n}} \\
&= \frac{1}{2} f(a/k, k, -1, q) + f\left(a, k^2, aq, q^2\right).
\end{aligned}$$

Here the first equality is by logarithmic differentiation (noting that $H(1) = 1$), the second is by simple combination/separation of some of the series, and the final equality follows from (3.4) and (3.6). Thus we have that

$$F(a,k,q) = \frac{1}{2} f\left(\frac{a}{k}, k, -1, q\right) + f\left(a, k^2, aq, q^2\right). \tag{3.10}$$

Upon replacing a with $1/a$ and k with $1/k$ in $F(a,k,q)$ and the Lambert series above, we get

$$F\left(\frac{1}{a}, \frac{1}{k}, q\right) = \frac{1}{2} f\left(\frac{k}{a}, \frac{1}{k}, -1, q\right) + f\left(\frac{1}{a}, \frac{1}{k^2}, \frac{1}{aq}, q^2\right) + \frac{q}{1-q} - \frac{\frac{k^2q}{a}}{1-\frac{k^2q}{a}}.$$

Thus,

$$F(a,k,q)-F\left(\frac{1}{a},\frac{1}{k},q\right)=\frac{1}{2}f\left(\frac{a}{k},k,-1,q\right)-\frac{1}{2}f\left(\frac{k}{a},\frac{1}{k},-1,q\right)$$
$$+f\left(a,k^2,aq,q^2\right)-f\left(\frac{1}{a},\frac{1}{k^2},\frac{1}{aq},q^2\right)-\frac{q}{1-q}+\frac{\frac{k^2q}{a}}{1-\frac{k^2q}{a}},$$

and (1.2) follows from (3.5) and (3.7), upon noting that

$$\frac{(a/k-k)(1+a)}{(1-a^2/k^2)(1-k^2)}+\frac{(a-k^2)\left(1-\frac{1}{aq}\right)\left(1-\frac{k^2}{q}\right)}{(1-a)(1-k^2)\left(1-\frac{1}{q}\right)\left(1-\frac{k^2}{aq}\right)}$$
$$-\frac{q}{1-q}+\frac{\frac{k^2q}{a}}{1-\frac{k^2q}{a}}=\frac{(a^2-k)(a-k^2)}{(1-a)(1-k)(a^2-k^2)}.$$

□

One easy implication is the following summation formula.

Corollary 1. *Let a and k be non-zero complex numbers and $|q|$ a complex number such that $|q|<\max\{1,|a/k|,|k/a|\}$ and suppose none of the denominators following vanish. Then*

$$\sum_{n=1}^{\infty}\frac{(1-kq^{2n})(q;q)_{n-1}\left(k^2/a,k,k/a;q\right)_n(qa;q^2)_n}{(1-k)\left(qa/k,kq,aq,q;q\right)_n(k^2q/a;q^2)_n}\left(\frac{-qa}{k}\right)^n \tag{3.11}$$
$$-\sum_{n=1}^{\infty}\frac{(1-q^{2n}/k)(q;q)_{n-1}\left(a/k^2,1/k,a/k;q\right)_n\left(q/a;q^2\right)_n}{(1-1/k)\left(qk/a,q/k,q/a,q;q\right)_n\left(aq/k^2;q^2\right)_n}\left(\frac{-qk}{a}\right)^n$$
$$=\frac{aq}{k^2}\frac{\left(aq,q/a,k^2/a,q^2a/k^2,k^2/q,q^3/k^2,q^2,q^2;q^2\right)_\infty}{\left(aq/k^2,k^2q/a,a,q^2/a,k^2,q^2/k^2,q,q;q^2\right)_\infty}$$
$$-\frac{a}{k}\frac{\left(k^2/a,qa/k^2,-a,-q/a;q\right)_\infty(q^2,q^2;q^2)_\infty}{\left(k^2,q^2/k^2,a^2/k^2,q^2k^2/a^2;q^2\right)_\infty}+\frac{(a^2-k)(a-k^2)}{(1-a)(1-k)(a^2-k^2)}.$$

Proof. Insert the"trivial" Bailey pair

$$\alpha_n(a,q)=\begin{cases}1 & n=0,\\ 0, & n>0,\end{cases} \tag{3.12}$$
$$\beta_n(a,q)=\frac{(k,k/a;q)_n}{(aq,q;q)_n}.$$

into (1.3). □

Inserting the unit WP-Bailey pair,

$$\alpha_n(a,k)=\frac{(q\sqrt{a},-q\sqrt{a},a,a/k;q)_n}{(\sqrt{a},-\sqrt{a},q,kq;q)_n}\left(\frac{k}{a}\right)^n, \tag{3.13}$$
$$\beta_n(a,k)=\begin{cases}1 & n=0,\\ 0, & n>1,\end{cases}$$

likewise leads to a four-term summation formula, with the same right side as (3.11).

4. New θ-Function Identities and new transformations for basic hypergeometric series.

We consider some other implications of (1.2) and (1.3).

Corollary 2. *Let $|q|<1$ and $(\alpha_n(a,k,q),\beta_n(a,k,q))$ be a WP-Bailey pair. Then*

$$\begin{aligned}\sum_{n=1}^{\infty}&\frac{(1-kq^{2n})(q;q)_{n-1}\left(\frac{k^2}{q};q\right)_n(q^2;q^2)_n}{(1-k)\left(\frac{q^2}{k},kq;q\right)_n(k^2;q^2)_n}\left(\frac{-q^2}{k}\right)^n\beta_n(q,k,q)\\ &-\sum_{n=1}^{\infty}\frac{(q^2;q^2)_{n-1}\left(\frac{k^2}{q};q^2\right)_n q^{4n}\alpha_{2n}(q,k,q)}{\left(\frac{q^4}{k^2},q^3;q^2\right)_n k^{2n}}+\frac{\left(k^2-q\right)\left(k-q^2\right)}{(1-k)(1-q)(k^2-q^2)}\\ &=k\frac{(k^2q,q/k^2,q^2,q^2;q^2)_\infty}{(k^2,q^2/k^2,q,q;q^2)_\infty}\left(1+\sum_{n=0}^{\infty}\frac{\left(\frac{k^2}{q^2},\frac{1}{q};q^2\right)_{n+1}q^{4n+4}\alpha_{2n+1}(q,k,q)}{\left(\frac{q^3}{k^2},q^2;q^2\right)_{n+1}k^{2n+2}}\right).\end{aligned} \tag{4.1}$$

Proof. From (3.9) and (3.10),

$$\begin{aligned}F(q,k,q)&=\sum_{n=1}^{\infty}\frac{kq^n}{1-k^2q^{2n}}-\sum_{n=1}^{\infty}\frac{q^{n+1}/k}{1-q^{2n+2}/k^2}+\frac{q}{1-q}-\frac{k^2}{1-k^2}\\ &=\sum_{n=1}^{\infty}\frac{kq^n}{1-k^2q^{2n}}-\sum_{n=1}^{\infty}\frac{q^n/qk}{1-q^{2n}/q^2k^2}\\ &\qquad+\frac{1/k}{1-1/k^2}+\frac{q/k}{1-q^2/k^2}+\frac{q}{1-q}-\frac{k^2}{1-k^2}\\ &=\sum_{n=1}^{\infty}\frac{kq^n}{1-k^2q^{2n}}-\sum_{n=1}^{\infty}\frac{q^n/qk}{1-q^{2n}/q^2k^2}+\frac{\left(k^2-q\right)\left(q^2-k\right)}{(1-k)(1-q)(k^2-q^2)}\\ &=k\frac{(k^2q,q/k^2,q^2,q^2;q^2)_\infty}{(k^2,q^2/k^2,q,q;q^2)_\infty}+\frac{\left(k^2-q\right)\left(q^2-k\right)}{(1-k)(1-q)(k^2-q^2)},\end{aligned} \tag{4.2}$$

where an identity of Ramanujan ([4, Chapter. 17, page 116, Equation (8.5)]) is used to combine the two Lambert series to give the infinite product.

Upon using (1.2) to substitute for $F(q,k,q)$ above, we get the identity

$$\sum_{n=1}^{\infty} \frac{(1-kq^{2n})(q;q)_{n-1}\left(\frac{k^2}{q};q\right)_n (q^2;q^2)_n}{(1-k)\left(\frac{q^2}{k},kq;q\right)_n (k^2;q^2)_n} \left(\frac{-q^2}{k}\right)^n \beta_n(q,k) \tag{4.3}$$

$$-\sum_{n=1}^{\infty} \frac{(q^2;q^2)_{n-1}\left(\frac{k^2}{q};q^2\right)_n}{\left(\frac{q^4}{k^2},q^3;q^2\right)_n} \left(\frac{q^2}{k}\right)^{2n} \alpha_{2n}(q,k)$$

$$+\frac{\left(\frac{k^2}{q},\frac{q^5}{k^2},q^4,q^2;q^2\right)_\infty}{\left(k^2,\frac{q^4}{k^2},q^3,q;q^2\right)_\infty} \sum_{n=0}^{\infty} \frac{(k^2,q;q^2)_n}{\left(\frac{q^5}{k^2},q^4;q^2\right)_n} \left(\frac{q^2}{k}\right)^{2n+1} \alpha_{2n+1}(q,k)$$

$$= k\frac{(k^2q,q/k^2,q^2,q^2;q^2)_\infty}{(k^2,q^2/k^2,q,q;q^2)_\infty} + \frac{\left(k^2-q\right)\left(q^2-k\right)}{(1-k)(1-q)(k^2-q^2)}.$$

This last identity gives (4.1), after some simple manipulations. □

Remark: We note in passing that if $R(q)$ denotes the left side of (4.1), and $S(q)$ denotes the infinite series following the infinite product on the right side of (4.1), then

$$\frac{(1-k^2)R(q)}{kS(q)}$$

is invariant under the transformation $k \to 1/k$.

Let

$$M(k,q) := \frac{(k^2q,q/k^2,q^2,q^2;q^2)_\infty}{(k^2,q^2/k^2,q,q;q^2)_\infty}, \tag{4.4}$$

the infinite product at (4.1) above. We also note that many of the theta functions investigated by Ramanujan and others are expressible in terms of $M(k,q)$, so that inserting the WP-Bailey pair

$$\alpha_n(a,k) = \begin{cases} 0, & \text{if } n \text{ is odd,} \\ \frac{\left(q^2\sqrt{a},-q^2\sqrt{a},a,a^2/k^2;q^2\right)_{n/2}}{\left(\sqrt{a},-\sqrt{a},q^2,q^2k^2/a;q^2\right)_{n/2}}\left(\frac{k}{a}\right)^n, & \text{if } n \text{ is even,} \end{cases} \tag{4.5}$$

$$\beta_n(a,k) = \frac{\left(k,\, k\sqrt{q/a},\, -k\sqrt{q/a},\, a/k;\, q\right)_n}{(\sqrt{aq},\, -\sqrt{aq},\, qk^2/a,\, q;\, q)_n}\left(\frac{-k}{a}\right)^n,$$

with $a = q$ and the appropriate choice k (which we give below), in (4.1) provides representations of these theta functions in terms of basic hypergeometric series.

For example, recall the function

$$a(q) := \sum_{m,n=-\infty}^{\infty} q^{m^2+mn+n^2}. \tag{4.6}$$

This series was studied in [6], where it was shown that

$$a^3(q) = b^3(q) + c^3(q),$$

where $b(q) = \sum_{m,n=-\infty}^{\infty} \omega^{m-n} q^{m^2+mn+n^2}$, $\omega = exp(2\pi i/3)$, and $c(q) = \sum_{m,n=-\infty}^{\infty} q^{(m+1/3)^2+(m+1/3)(n+1/3)+(n+1/3)^2}$. The series $a(q)$ was also studied by Ramanujan, who showed (**Entry 18.2.8** of Ramanujan's Lost Notebook - see [2, page 402]) that

$$a(q) = 1 + 6\sum_{n=1}^{\infty} \frac{q^{-2}q^{3n}}{1-q^{-2}q^{3n}} - 6\sum_{n=1}^{\infty} \frac{q^{-1}q^{3n}}{1-q^{-1}q^{3n}}.$$

In [5, Equation (6.3), page 116] the author showed that

$$a(q) - a(q^2) = 6q\frac{(q, q^5, q^6, q^6; q^6)_\infty}{(q^2, q^4, q^3, q^3; q^6)_\infty} = 6q\frac{\psi^3(q^3)}{\psi(q)}. \tag{4.7}$$

This result may be derived from (3.10) by replacing q with q^3, a with $1/q^3$ and k with $1/q^2$, then inserting the unit WP-Bailey pair (3.13) and finally using (4.7) to combine the resulting Lambert series. Notice that the last identity gives that $a(q) - a(q^2) = 6qM(q, q^3)$, with the implications given by (4.1) noted above.

As a second example, consider the function $q\psi(q^2)\psi(q^6)$. Ramanujan showed (see **Entry 3** (i), Chapter 19, page 223 of [4]) that,

$$q\psi(q^2)\psi(q^6) = \sum_{n=1}^{\infty} \frac{q^{6n-5}}{1-q^{12n-10}} - \sum_{n=1}^{\infty} \frac{q^{6n-1}}{1-q^{12n-2}}. \tag{4.8}$$

With regard to (4.1), we note that $q\psi(q^2)\psi(q^6) = qM(q, q^6)$.

Thirdly, recall the function

$$\phi(q) := \sum_{n=-\infty}^{\infty} q^{n^2} = (-q, -q, q^2; q^2)_\infty. \tag{4.9}$$

Ramanujan (**Entry 8 (i)** in chapter 17 of [4]) showed that

$$\phi(q)^2 = 1 + 4\sum_{n=1}^{\infty} \frac{q^{4n-3}}{1-q^{4n-3}} - 4\sum_{n=1}^{\infty} \frac{q^{4n-1}}{1-q^{4n-1}},$$

and is easy to check that $\phi(q)^2 = 2M(i, q)$, where $i^2 = -1$.

4.1. **A variation of** (4.1)**.** It is not difficult to see that if we set $a = q^{2t+1}$, where t is an integer, in (3.9) and (3.10), then the Lambert series combine in essentially the same way as they did at (4.2), with a different finite sum of terms left over from combining the Lambert series that essentially cancel each other, (the particular sum depending on the choice of t). Corollary 2 follows from the choice $t = 0$, but a similar result will follow from other choices for t. If we let t be a negative integer, then inserting the usual WP-Bailey pairs in the resulting identity will, in most cases, give trivial results (for most WP-Bailey pairs, either the α_n or the β_n contains a $(a;q)_n$ factor, or some similar factor that will vanish for all n large enough when a is a negative power of q). However, there are two WP-Bailey pairs which give non-trivial results for the choice $a = q^{-1}$, and we consider those results next.

The proof of the following corollary is virtually identical to that of Corollary 3 (except, as mentioned, a is set equal to $1/q$) and so is omitted.

Corollary 3. *Let $|q|<1$ and $(\alpha_n(a,k,q),\beta_n(a,k,q))$ be a WP-Bailey pair. Define $\beta_n^*(1/q,k,q)=\lim_{a\to 1/q}(1-aq)\beta_n(a,k,q)$, if the limit exists. Then, assuming all series converge,*

$$\sum_{n=1}^{\infty}\frac{(1-kq^{2n})(q;q)_{n-1}\left(k^2q;q\right)_n(q^2;q^2)_{n-1}}{(1-k)\left(\frac{1}{k},kq;q\right)_n(k^2q^2;q^2)_n}\left(\frac{-1}{k}\right)^n\beta_n^*(1/q,k,q) \tag{4.10}$$
$$-\sum_{n=1}^{\infty}\frac{(q^2;q^2)_{n-1}\left(k^2q;q^2\right)_n\alpha_{2n}(1/q,k,q)}{\left(\frac{1}{k^2},q;q^2\right)_n k^{2n}}$$
$$=k\frac{(k^2q,q/k^2,q^2,q^2;q^2)_\infty}{(k^2,q^2/k^2,q,q;q^2)_\infty}\left(1+\sum_{n=0}^{\infty}\frac{\left(k^2q^2,q;q^2\right)_n\alpha_{2n+1}(1/q,k,q)}{\left(\frac{q}{k^2},q^2;q^2\right)_n k^{2n}}\right).$$

Inserting the WP-Bailey pair

$$\alpha_n^{(1)}(a,k)=\frac{(qa^2/k^2;q)_n}{(q,q)_n}\left(\frac{k}{a}\right)^n, \tag{4.11}$$
$$\beta_n^{(1)}(a,k)=\frac{(qa/k,k;q)_n}{(k^2/a,q,q)_n}\frac{(k^2/a;q)_{2n}}{(aq,q)_{2n}},$$

leads, for $|k|>1$, to the identity

$$\sum_{n=1}^{\infty}\frac{(1-kq^{2n})\left(k^2q;q\right)_n}{(1-q^n)(1-kq^n)\left(q;q^2\right)_n}\left(\frac{-1}{k}\right)^n-\sum_{n=1}^{\infty}\frac{\left(k^2q,\frac{1}{k^2q};q^2\right)_n q^{2n}}{(1-q^{2n})\left(q,q;q^2\right)_n} \tag{4.12}$$
$$=k\frac{(k^2q,q/k^2,q^2,q^2;q^2)_\infty}{(k^2,q^2/k^2,q,q;q^2)_\infty}\left(1+\sum_{n=0}^{\infty}\frac{\left(1-\frac{1}{k^2q}\right)\left(k^2q^2,\frac{1}{k^2};q^2\right)_n q^{2n+1}}{(1-q^{2n+1})\left(q^2,q^2;q^2\right)_n}\right).$$

Similarly, inserting the pair at (4.5) gives the identity

$$\sum_{n=1}^{\infty}\frac{(1-kq^{2n})(q;q)_{n-1}\left(k^2q,k,\frac{1}{kq};q\right)_n q^n}{\left(q,k^2q^2,\frac{1}{k},kq;q^2\right)_n} \tag{4.13}$$
$$-\sum_{n=1}^{\infty}\frac{(1-q^{4n-1})(q^2;q^2)_{n-1}\left(k^2q,\frac{1}{k^2q^2},\frac{1}{q};q^2\right)_n q^{2n}}{(1-q^{-1})\left(k^2q^3,\frac{1}{k^2},q,q^2;q^2\right)_n}$$
$$=k\frac{(k^2q,q/k^2,q^2,q^2;q^2)_\infty}{(k^2,q^2/k^2,q,q;q^2)_\infty}.$$

The next result also follows from (1.2), and expresses a general sum involving an arbitrary WP-Bailey pair in terms of Lambert series.

Corollary 4. *Let $(\alpha_n(a,k),\beta_n(a,k))$ be a WP-Bailey pair. Then*

$$\begin{aligned}(4.14)\quad &\sum_{n=1}^{\infty}\frac{(1-kq^{2n})(q,q;q)_{n-1}(qk^2;q^2)_n}{(1-k)\,(kq,kq;q)_n\,(q;q^2)_n}\,(-qk)^n\,\beta_n(k^2,k)\\
&-\sum_{n=1}^{\infty}\frac{(q^2,q^2;q^2)_{n-1}}{(q^2k^2,q^2k^2;q^2)_n}\,(qk)^{2n}\,\alpha_{2n}(k^2,k)\\
&+\frac{\left(q^3k^2,q^3k^2,q^2,q^2;q^2\right)_\infty}{(q^2k^2,q^2k^2,q,q;q^2)_\infty}\sum_{n=0}^{\infty}\frac{\left(q,q;q^2\right)_n}{(q^3k^2,q^3k^2;q^2)_n}\,(qk)^{2n+1}\,\alpha_{2n+1}(k^2,k)\\
&=\sum_{n=1}^{\infty}\frac{nq^n}{1-q^{2n}}+\sum_{n=1}^{\infty}\frac{nk^{2n}q^{2n}}{1-q^{2n}}-\sum_{n=1}^{\infty}\frac{nk^nq^n}{1-q^n}.\end{aligned}$$

Proof. Define

$$\begin{aligned}(4.15)\quad G(k,q):=&\sum_{n=1}^{\infty}\frac{(1-kq^{2n})(q,q;q)_{n-1}(qk^2;q^2)_n}{(1-k)\,(kq,kq;q)_n\,(q;q^2)_n}\,(-qk)^n\,\beta_n(k^2,k)\\
&-\sum_{n=1}^{\infty}\frac{(q^2,q^2;q^2)_{n-1}}{(q^2k^2,q^2k^2;q^2)_n}\,(qk)^{2n}\,\alpha_{2n}(k^2,k)\\
&+\frac{\left(q^3k^2,q^3k^2,q^2,q^2;q^2\right)_\infty}{(q^2k^2,q^2k^2,q,q;q^2)_\infty}\sum_{n=0}^{\infty}\frac{\left(q,q;q^2\right)_n}{(q^3k^2,q^3k^2;q^2)_n}\,(qk)^{2n+1}\,\alpha_{2n+1}(k^2,k).\end{aligned}$$

From (1.2), it can be seen that $F(k^2,k,q)=0$ and that

(4.16)
$$G(k,q)=\lim_{a\to k^2}\frac{F(a,k,q)}{1-k^2/a}=\lim_{a\to k^2}\frac{a(F(a,k,q)-F(k^2,k,q))}{a-k^2}=k^2M'(k^2),$$

where $M(a):=F(a,k,q)$. The result follows from the representation of $M(a)$ as a sum of Lambert series on the right side of (3.10), after some simple algebraic manipulations, and after using the identity

$$\sum_{n=1}^{\infty}\frac{xq^n}{(1-xq^n)^2}=\sum_{n=1}^{\infty}\frac{nx^nq^n}{1-q^n}$$

a number of times. □

The identity at (4.14) extends a result by the first author in a previous paper [11, Equation (7.3)], where the identity which follows from (4.14) upon inserting the trivial WP-Bailey pair (3.12) was proven.

Upon replacing q with q^2 and k with $1/q$ in (4.14) and inserting the unit WP-Bailey pair (3.13), we derive Ramanujan's identity (**Example** (iii) on page 139 of [4])

$$q\psi^4(q^2)=\sum_{k=0}^{\infty}\frac{(2k+1)q^{2k+1}}{1-q^{4k+2}}. \tag{4.17}$$

where $\psi(q)$ is defined at (1.4). The same substitutions in (4.14) followed by the insertion of the trivial WP-Bailey pair (3.12) gives the identity ([11, Corollary14])

$$\sum_{n=0}^{\infty} \frac{(1-q^{4n+3})(q^2;q^2)_n(q^4;q^4)_n(-q)^n}{(1-q^{2n+1})(q^2;q^2)_{n+1}(q^2;q^4)_{n+1}} = \psi^4(q^2). \tag{4.18}$$

It is possible to derive a more general identity involving the function $q\psi^4(q^2)$.

Corollary 5. *Let $(\alpha_n(a,k,q), \beta_n(a,k,q))$ be a WP-Bailey pair. Then*

$$\begin{aligned}
&\sum_{n=1}^{\infty} \frac{1+q^{2n}}{2} \frac{(q,q;q)_{n-1}q^n}{(-q,-q;q)_n} \beta_n(1,-1,q) \\
&\quad - \sum_{n=1}^{\infty} \frac{1+q^{2n}}{2} \frac{(-q,-q;-q)_{n-1}\,(-q)^n}{(q,q;-q)_n} \beta_n(1,-1,-q) \\
&\quad - \sum_{n=1}^{\infty} \frac{q^n}{(1-q^n)^2}\,\alpha_n(1,-1,q) + \sum_{n=1}^{\infty} \frac{(-q)^n}{(1-(-q)^n)^2}\,\alpha_n(1,-1,-q) \\
&\qquad = 4q\psi^4(q^2).
\end{aligned} \tag{4.19}$$

$$\begin{aligned}
&\sum_{n=1}^{\infty} \frac{1+q^{2n}}{1-q^{2n}} \frac{(q;q)_{n-1}q^{n(n+1)/2}}{(-q;q)_n} - \sum_{n=1}^{\infty} \frac{1+q^{2n}}{1-q^{2n}} \frac{(-q;-q)_{n-1}(-q)^{n(n+1)/2}}{(q;-q)_n} + 2q\times \\
&\sum_{n=1}^{\infty} q^{8n^2-4n} \left(\frac{\left(q^{8n-2}+3\right)q^{6n-2}}{(1-q^{8n-2})^2} - \frac{\left(q^{4n-2}+1\right)q^{-2n}}{(1-q^{4n-2})^2} + \frac{\left(3q^{8n-6}+1\right)q^{2-6n}}{(1-q^{8n-6})^2} \right) \\
&\qquad = 4q\psi^4(q^2).
\end{aligned} \tag{4.20}$$

Proof. From (4.15), the left side of (4.19) is $G(-1,q) - G(-1,-q)$. On the other hand, from (4.14),

$$\begin{aligned}
G(-1,q) &= \sum_{n=1}^{\infty} \frac{nq^n}{1-q^{2n}} + \sum_{n=1}^{\infty} \frac{nq^{2n}}{1-q^{2n}} - \sum_{n=1}^{\infty} \frac{n(-1)^n q^n}{1-q^n} \\
&= \sum_{n=1}^{\infty} \frac{nq^n}{1-q^n} - \sum_{n=1}^{\infty} \frac{n(-1)^n q^n}{1-q^n} \\
&= 2\sum_{n=1}^{\infty} \frac{(2n-1)q^{2n-1}}{1-q^{2n-1}} \\
&= 2\sum_{n=1}^{\infty} \frac{(2n-1)q^{2n-1}}{1-q^{4n-2}} + 2\sum_{n=1}^{\infty} \frac{(2n-1)q^{4n-2}}{1-q^{4n-2}}.
\end{aligned}$$

Thus

$$G(-1,q) - G(-1,-q) = 4\sum_{n=1}^{\infty} \frac{(2n-1)q^{2n-1}}{1-q^{4n-2}},$$

and (4.19) follows from (4.17).

For (4.20), we start with Singh's WP-Bailey pair [14],

$$\begin{aligned}\alpha_n(a,k,q) &= \frac{(q\sqrt{a},-q\sqrt{a},a,\rho_1,\rho_2,a^2q/k\rho_1\rho_2;q)_n}{(\sqrt{a},-\sqrt{a},q,aq/\rho_1,aq/\rho_2,k\rho_1\rho_2/a;q)_n}\left(\frac{k}{a}\right)^n,\\ \beta_n(a,k,q) &= \frac{(k\rho_1/a,k\rho_2/a,k,aq/\rho_1\rho_2;q)_n}{(aq/\rho_1,aq/\rho_2,k\rho_1\rho_2/a,q;q)_n},\end{aligned}$$

set $a=1$,$k=-1$ and let $\rho_1,\rho_2\to\infty$ to get the pair

$$\begin{aligned}\alpha_n(1,-1,q) &= (1+q^n)(-1)^n q^{n(n-1)/2},\\ \beta_n(1,-1,q) &= \frac{(-1;q)_n q^{n(n-1)/2}}{(q;q)_n}.\end{aligned}$$

The first two series in (4.20) come directly from inserting the expressions for $\beta_n(1,-1,q)$ and $\beta_n(1,-1,-q)$ in the first two series in (4.19). The third series in (4.20) comes inserting the expressions for $\alpha_n(1,-1,q)$ and $\alpha_n(1,-1,-q)$ in the last two series in (4.19), then replacing n, in turn, with $4n$, $4n-1$, $4n-2$ and $4n-3$, and then combining each pair of series into a single series, and finally combining the three surviving series together into one series. □

Remark: It is not difficult to see that a similar consideration of

$$\lim_{a\to k^2}\frac{F(a,k,q)-F(1/a,1/k,k)}{1-k^2/a}$$

at (1.3) gives the following result.

Corollary 6. *Let $(\alpha_n(a,k),\beta_n(a,k))$ be a WP-Bailey pair and let $G(k,q)$ be as defined at* (4.15)*. Then*

$$\begin{aligned}(4.21)\quad & G(k,q)+G(1/k,q)\\ &= \sum_{n=1}^{\infty}\frac{2nq^n}{1-q^{2n}}+\sum_{n=1}^{\infty}\frac{nk^{2n}q^{2n}}{1-q^{2n}}+\sum_{n=1}^{\infty}\frac{nq^{2n}/k^{2n}}{1-q^{2n}}-\sum_{n=1}^{\infty}\frac{nk^nq^n}{1-q^n}-\sum_{n=1}^{\infty}\frac{nq^n/k^n}{1-q^n}\\ &= \frac{k(1-k^3)}{(1-k)(1-k^2)^2}-k\frac{(q,q,-k^2,-q/k^2;q)_\infty(q^2,q^2;q^2)_\infty}{(k^2,k^2,q^2/k^2,q^2/k^2;q^2)_\infty}\\ &\qquad -k^2\frac{(k^2q,k^2q,q/k^2,q/k^2,q^2,q^2,q^2,q^2;q^2)_\infty}{(k^2,k^2,q^2/k^2,q^2/k^2,q,q,q,q;q^2)_\infty}.\end{aligned}$$

The special case of the this identity that follows from inserting the trivial pair (3.12) into the term $G(k,q)+G(1/k,q)$ also follows from Corollary 12 in [11], upon dividing the identity there by $1-b$ and then letting $b\to 1$.

As well as implying some of the known identities relating Lambert series and infinite products, the term $G(k,q)+G(1/k,q)$ in (4.21) also provides an additional expression for the Lambert series or the infinite product in terms of series involving an arbitrary WP-Bailey pair $(\alpha_n(a,k),\beta_n(a,k))$ (with $a=k^2$ and k specialized as required). We give two examples.

First recall that

$$\phi(q) := \sum_{n=-\infty}^{\infty} q^{n^2} = (-q;q^2)_\infty^2 (q^2;q^2)_\infty = \frac{(-q,q^2;q^2)_\infty}{(q,-q^2;q^2)_\infty}.$$

Corollary 7. *If $|q|<1$, then*

(4.22)

$$\begin{aligned} 1+8\sum_{n=1}^{\infty}\frac{nq^n}{1+(-q)^n} &= \phi^4(q) \\ &= 1+4\sum_{n=1}^{\infty}\frac{(1-iq^{2n})(q,q;q)_{n-1}(-1;q)_{2n}(-iq)^n}{(1-i)(iq,iq;q)_n(q;q)_{2n}} \\ &\quad +4\sum_{n=1}^{\infty}\frac{(1+iq^{2n})(q,q;q)_{n-1}(-1;q)_{2n}(iq)^n}{(1+i)(-iq,-iq;q)_n(q;q)_{2n}}. \end{aligned}$$

Proof. Let $k=i$ in (4.21). It is not difficult to show that the Lambert series combine to give

$$2\sum_{n=1}^{\infty}\frac{2nq^{2n}}{1+q^{2n}}+2\sum_{n=1}^{\infty}\frac{(2n-1)q^{2n-1}}{1-q^{2n-1}} = 2\sum_{n=1}^{\infty}\frac{nq^n}{1+(-q)^n}.$$

It is also easy to see that the infinite product side simplifies to give

$$-\frac{1}{4}+\frac{1}{4}\phi^4(q).$$

For $G(i,q)+G(1/i,q)$, we use (4.15) with $k=i$ and insert the trivial WP-Bailey pair (3.12) (with $a=k^2$ and then $k=i$):

$$\begin{aligned} \alpha_n &= 0,\ n>0, \\ \beta_n &= \frac{(i,-i;q)_n}{(-q,q;q)_n} = \frac{(-1;q^2)_n}{(q^2;q^2)_n}. \end{aligned}$$

Multiply the resulting set of equalities by 4, add 1, and the identities at (4.22) follow. □

Remarks: The first equality at (4.22) is due to Jacobi [7]. Also, if other WP-Bailey pairs are used instead of the trivial pair above, then still further representations for $\phi^4(q)$ will result.

Corollary 8. *Let $\omega := \exp(2\pi i/3)$, let $\chi_0(n)$ denote the principal character modulo 3, and let $\psi(q)$ be as defined at (1.4). Then*

(4.23)

$$\begin{aligned} 9\sum_{n=1}^{\infty}\chi_0(n)\frac{nq^n}{1-q^{2n}} &= \frac{\psi^6(q)}{\psi^2(q^3)} - \frac{\psi^3(q^{1/2})\psi^3(-q^{1/2})}{\psi(q^{3/2})\psi(-q^{3/2})} \\ &= 3(1-\omega^2)\sum_{n=1}^{\infty}\frac{(1-\omega q^{2n})(q;q)_{n-1}(\omega^2 q;q^2)_n(-\omega q)^n}{(1-q^{3n})(\omega q;q)_n(q;q^2)_n} \end{aligned}$$

$$+3(1-\omega)\sum_{n=1}^{\infty}\frac{(1-\omega^2q^{2n})(q;q)_{n-1}(\omega q;q^2)_n(-\omega^2q)^n}{(1-q^{3n})(\omega^2q;q)_n(q;q^2)_n}.$$

Proof. The proof is similar to that of the previous corollary, except we set $k=\omega$ in (4.21). The Lambert series combine to give

$$3\sum_{n=1}^{\infty}\frac{(3n-1)q^{3n-1}}{1-q^{6n-2}}+3\sum_{n=1}^{\infty}\frac{(3n-2)q^{3n-2}}{1-q^{6n-4}}=3\sum_{n=1}^{\infty}\chi_0(n)\frac{nq^n}{1-q^{2n}}.$$

The infinite product side simplifies to give

$$\frac{1}{3}\frac{\psi^6(q)}{\psi^2(q^3)}-\frac{1}{3}\frac{\psi^3(q^{1/2})\psi^3(-q^{1/2})}{\psi(q^{3/2})\psi(-q^{3/2})}.$$

Once again, for $G(\omega,q)+G(1/\omega,q)$ we use (4.15), this time with $k=\omega$ and the trivial WP-Bailey pair (3.12) (with $a=k^2$ and then $k=\omega$):

$$\alpha_n=0,\ n>0,$$
$$\beta_n=\frac{(\omega,\omega^2;q)_n}{(\omega^2q,q;q)_n}.$$

Multiply the resulting set of equalities by 3, and the identities at (4.23) follow, after some simple manipulations of the expressions for $G(\omega,q)$ and $G(\omega^2,q)$. □

Remark: The Lambert series in the identity at (4.23) may also be represented in terms of the theta series $a(q)$ defined at(4.6), upon noting that

$$\sum_{n=1}^{\infty}\chi_0(n)\frac{nq^n}{1-q^{2n}}=\sum_{n=1}^{\infty}\chi_0(n)\frac{nq^n}{1-q^n}-\sum_{n=1}^{\infty}\chi_0(n)\frac{nq^{2n}}{1-q^{2n}},$$

and employing an identity of Ramanujan from the Lost Notebook, **Entry 18.2.9** (see [2], page 402), which states that

$$a^2(q)=1+12\sum_{n=1}^{\infty}\chi_0(n)\frac{nq^n}{1-q^n}.$$

References

[1] Andrews, George E. *Bailey's transform, lemma, chains and tree.* Special functions 2000: current perspective and future directions (Tempe, AZ), 1–22, NATO Sci. Ser. II Math. Phys. Chem., **30**, Kluwer Acad. Publ., Dordrecht, 2001.

[2] G.E. Andrews and B.C. Berndt, *Ramanujan's Lost Notebook, Part I*, Springer, 2005.

[3] Andrews, George E.; Lewis, Richard; Liu, Zhi-Guo *An identity relating a theta function to a sum of Lambert series.* Bull. London Math. Soc. **33** (2001), no. 1, 25–31.

[4] B. C. Berndt, *Ramanujan's Notebooks, Part III*, Springer-Verlag, New York, 1991.

[5] B. C. Berndt, *Ramanujan's Notebooks, Part V*, Springer-Verlag, New York, 1998.

[6] Borwein, J. M.; Borwein, P. B., *A cubic counterpart of Jacobi's identity and the AGM.* Trans. Amer. Math. Soc. **323** (1991), no. 2, 691–701.

[7] C. G. J. Jacobi *Fundamenta Nova Theoriae Functionum Ellipticarum*, Sumptibus fratrum Bornträgar, Regiomonti, 1829.

[8] Q. Liu; X. Ma *On the Characteristic Equation of Well-Poised Baily Chains* - To appear.

[9] J. Mc Laughlin, *Some new Transformations for Bailey pairs and WP-Bailey Pairs.* Cent. Eur. J. Math. **8** (2010), no. 3, 474-487.

[10] Mc Laughlin, James, *Some Further Transformations for WP-Bailey Pairs* - submitted

[11] Mc Laughlin, James, *New Summation Formulae for some q-products* - submitted

[12] Mc Laughlin, James; Zimmer, Peter *Some identities between basic hypergeometric series deriving from a new Bailey-type transformation.* J. Math. Anal. Appl. **345** (2008), no. 2, 670–677.

[13] Mc Laughlin, James; Zimmer, Peter. *General WP-Bailey Chains* - To appear in The Ramanujan Journal.

[14] Singh, U. B. *A note on a transformation of Bailey.* Quart. J. Math. Oxford Ser. (2) **45** (1994), no. 177, 111–116.

[15] Warnaar, S. O. *Extensions of the well-poised and elliptic well-poised Bailey lemma.* Indag. Math. (N.S.) **14** (2003), no. 3-4, 571–588.

Mathematics Department, 25 University Avenue, West Chester University, West Chester, PA 19383
E-mail address: jmclaughl@wcupa.edu

Mathematics Department, 25 University Avenue, West Chester University, West Chester, PA 19383
E-mail address: pzimmer@wcupa.edu