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A Hardy-Ramanujan-Rademacher-type formula for (r, s)-regular partitions

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Abstract Let $p_{r,s}(n)$ denote the number of partitions of a positive integer n into parts containing no multiples of r or s, where $r > 1$ and $s > 1$ are square-free, relatively prime integers. We use classical methods to derive a Hardy-Ramanujan-Rademacher-type infinite series for $p_{r,s}(n)$.

Keywords q-series · partitions · circle-method · Hardy-Ramanujan-Rademacher

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1 Introduction

A *partition* of a positive integer n is a representation of n as a sum of positive integers, where the order of the summands does not matter. We use $p(n)$ to denote the number of partitions of n, so that, for example, $p(4) = 5$, since 4 may be represented as 4, $3+1$, $2+2$, $2+1+1$ and $1+1+1+1$. The function $p(n)$ increases rapidly with n, and it is difficult to compute $p(n)$ directly for large n.

Rademacher [23], by slightly modifying earlier work of Hardy and Ramanujan [13], derived a remarkable infinite series for $p(n)$. To describe this series

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we need some notation. Recall that the Dedekind sum $s(e, f)$ is defined by

$$s(e, f) := \sum_{r=1}^{f-1} \frac{r}{f} \left(\frac{er}{f} - \left\lfloor \frac{er}{f} \right\rfloor - \frac{1}{2} \right),$$

and for ease of notation, we use $\omega(e, f)$ to denote $\exp(\pi i\, s(e, f))$, and for a positive integer k, set

$$A_k(n) := \sum_{\substack{0 \le h < k \\ (h,k)=1}} \omega(h, k) e^{-2\pi i n h/k}.$$

Recall also that $I_\nu(z)$ denotes the modified Bessel function of the first kind.

Theorem 1 *(Rademacher) If n is a positive integer, then*

$$p(n) = \frac{2\pi}{(24n-1)^{3/4}} \sum_{k=1}^{\infty} \frac{A_k(n)}{k} I_{3/2} \left(\frac{\pi}{k} \sqrt{\frac{2}{3}\left(n - \frac{1}{24}\right)} \right). \tag{1.1}$$

Rademacher's series converges incredibly fast. For example,

$$p(500) = 2,300,165,032,574,323,995,027,$$

and yet six terms of the series are sufficient to get within 0.5 of $p(500)$. The idea of course is that if a partial sum is known to be within 0.5 of the value of the series, then the nearest integer gives the exact value of $p(n)$.

Since the publication of Rademacher's paper [23], a number of authors have found series similar to (1.1) for certain restricted partition functions. Lehner [19] found such series for $p_1(n)$ and $p_2(n)$, the number of partitions of n into parts $\equiv \pm 1 \pmod 5$ and $\equiv \pm 2 \pmod 5$ respectively, and this was extended by Livingood [20] to series for $p_1(n), \dots, p_{(q-1)/2}(n)$, the number of partitions into parts $\equiv \pm 1 \pmod q$, $\equiv \pm 2 \pmod q$, $\dots$, $\equiv \pm(q-1)/2 \pmod q$ respectively, where $q > 3$ is an odd prime. Hua [14] derived a Rademacher-type series for $p_O(n)$, the number of partitions of n into odd parts.

Let $q \ge 3$ be an odd prime and $a = \{a_1, a_2, \dots, a_r\}$ be a set of distinct integers satisfying $1 \le a_i \le (q-1)/2$. Hagis [5] gave a Hardy-Ramanujan-Rademacher-type series (H.R.R. series) for $p_a(n)$, the number of partitions of n into parts $\equiv \pm a_i \pmod q$. In a subsequent series of papers [6–12], Hagis also developed similar series for other restricted partition functions (into odd parts, odd distinct parts, no part repeated more than t times, *etc.*).

Niven [21] gave a H.R.R. series for $p_{2,3}(n)$, the number of partitions of n into parts containing no multiples of 2 or 3. In a similar vein, Haberzetle [4] gave a series for $p_{q_1,q_2}(n)$, the number of partitions of n into parts containing no multiples of q_1 or q_2, where q_1 and q_2 are distinct primes such that $24|(q_1 - 1)(q_2 - 1)$.

Iseki [15–17] derived H.R.R. series that, amongst other results, extended the result of Livingood [20] cited above from a prime q to a composite integer M, and also extended the results of Niven [21] and Haberzetle [4], by finding a H.R.R. series for $p_M(n)$, the number of partitions of n into parts relatively prime to a square-free positive integer M.

Sastri *et al.* [22,24,25] derived a number of H.R.R. series which, amongst other results, extended the result of Hagis cited above from a prime q to an arbitrary positive integer m.

More recently, Sills [26–28] has partly automated the process of finding H.R.R. series for restricted partition functions, and aided by the use of the computer algebra system *Mathematica*, has found many new such series, including ones for restricted partition functions represented by various identities of Rogers-Ramanujan type.

When $r>1$ and $s>1$ are relatively prime integers, let $p_{r,s}(n)$ denote the number of partitions of n into parts containing no multiples of r or s. We say that such a partition of an integer n is (r,s)*-regular*. In the present paper we give a H.R.R. series for $p_{r,s}(n)$ when r and s are square-free. We note that this result includes those Niven [21] and Haberzetle [4] as special cases.

We now state our result explicitly. Define

$$F(\tau)=\frac{1}{\prod_{k=1}^{\infty}(1-e^{2\pi ik\tau})},$$

and denote by $H_{i,j}$ a solution to the congruence $iH_{i,j}\equiv -1 \pmod{j}$, and for consistency of notation below, set $H_{0,1}=0$. For integers k, r and s, let $r_k:=\gcd(r,k)$ and $s_k:=\gcd(s,k)$ and, for ease of notation, set

$$R:=\frac{(r-1)(s-1)}{24},\qquad \delta_k:=\frac{(r/r_k-r_k)(s/s_k-s_k)}{24}. \tag{1.2}$$

Our result may be stated as follows.

Theorem 2 *Let $r>1$ and $s>1$ be square-free relatively prime integers. For a positive integer k and non-negative integer h with $(h,k)=1$, define the sequence $\{c_m(h,k)\}$ by*

$$\frac{F\left(\frac{H_{h,k}}{k}+\frac{i}{z}\right)F\left(\frac{H_{hrs/(r_ks_k),k/(r_ks_k)}}{k/(r_ks_k)}+\frac{ir_k^2s_k^2}{rsz}\right)}{F\left(\frac{H_{hr/r_k,k/r_k}}{k/r_k}+\frac{ir_k^2}{rz}\right)F\left(\frac{H_{hs/s_k,k/s_k}}{k/s_k}+\frac{is_k^2}{sz}\right)}:=\sum_{m=0}^{\infty}c_m(h,k)\exp\left(\frac{-2\pi mr_ks_k}{rsz}\right).$$

If $n>R$, then

$$p_{r,s}(n)=\sum_{k=1}^{\infty}\sum_{m=0}^{\lfloor\delta_k\rfloor}\frac{2\pi A_{k,m}(n)}{k}\sqrt{\frac{r_ks_k(\delta_k-m)}{rs(n-R)}}I_1\left(\frac{4\pi}{k}\sqrt{\frac{r_ks_k}{rs}(\delta_k-m)(n-R)}\right),$$

where

$$A_{k,m}(n):=\sum_{\substack{h=0\\(h,k)=1}}^{k-1}\frac{\omega(h,k)\omega(hrs/(r_ks_k),k/(r_ks_k))}{\omega(hr/r_k,k/r_k)\omega(hs/s_k,k/s_k)}c_m(h,k)\exp\left(\frac{-2\pi inh}{k}\right).$$

The method of proof follows to a large extent the method used by previous authors to derive similar convergent series for other partition functions. In section 2, the Cauchy Residue Theorem is applied to the generating function for the sequence $p_{r,s}(n)$, and a change of variable is then applied to convert the path of integration to the line segment $[i, i+1]$. Next, this line segment is deformed to follow the path along the top of a collection of Ford circles, after which another change of variable transforms the arc along the top of each Ford circle to an arc along the circle in the complex plain with center $1/2$ and radius $1/2$. Next, the transformation formula for the Dedekind eta function $\eta(\tau)$ is used to transform the integrand into a form whose properties can be exploited to derive the final series stated in Theorem 2. Each transformed infinite product is expanded in a series, which is broken into an initial finite part (which eventually leads to the series of the theorem) and a tail, whose contribution is shown to be negligible.

The path of integration for each of the terms coming from the tail of the series mentioned above is divided into three arcs. In section 3, Kloosterman sum estimates are developed, which are used in section 4 to get error bounds on the integrals along the three arcs for each term in the tail. This shows that these error terms go to zero as $N \to \infty$, where N is the order of the Farey sequence giving rise to the collection of Ford circles.

In section 5, the arcs of integration along the circle with center $1/2$ and radius $1/2$ for the main terms are replaced with a new path along the entire circle. It is shown that the contributions from the additional arcs also go to zero as $N \to \infty$, where N is as in the paragraph above. Two other changes of variable and an application of an integral formula for modified Bessel functions of the first kind lead the final result.

Remark: With the notation for $F(\tau)$ as above and for $\eta(\tau)$ as below, the generating functions

$$e^{-\pi i(r-1)(s-1))\tau/12}\frac{F(\tau)F(rs\tau)}{F(r\tau)F(s\tau)} = \frac{\eta(r\tau)\eta(s\tau)}{\eta(\tau)\eta(rs\tau)}$$

are weight-zero modular forms, so that the general theorem of Bringmann and Ono [3] could in theory be used to derive our series for $p_{r,s}(n)$. However, we prefer to employ the Hardy-Ramanujan-Rademacher method.

2 Initial transformations

Let

$$G(x) = \sum_{n=0}^{\infty} p_{r,s}(n)x^n = \frac{(x^r; x^r)_\infty (x^s; x^s)_\infty}{(x; x)_\infty (x^{rs}; x^{rs})_\infty} \tag{2.1}$$

denote the generating function for the sequence $\{p_{r,s}(n)\}$. By the Cauchy Residue Theorem,

$$p_{r,s}(n) = \frac{1}{2\pi i}\int_C \frac{G(x)}{x^{n+1}}dx,$$

where C is any positively oriented simple closed curve inside the unit circle containing the origin. As usual, we start by taking C to be the circle centered at the origin with radius $e^{-2\pi}$, and make the change of variable $x = e^{2\pi i\tau}$ to get

$$p_{r,s}(n) = \int_i^{i+1} G(e^{2\pi i\tau})e^{-2\pi i n\tau}d\tau.$$

We follow Rademacher by deforming the path of integration so that it traces the upper arcs of the collection of Ford circles

$$\left\{C_{h,k} : \frac{h}{k} \in \mathcal{F}_N\right\},$$

where $C_{h,k}$ is the circle with center $h/k + i/(2k^2)$ and radius $1/(2k^2)$, and $\mathcal{F}_N$ is the set of Farey fractions of order N. We denote the part of the path that is an arc of the circle $C_{h,k}$ by $\gamma(h,k)$. Thus

$$p_{r,s}(n) = \sum_{k=1}^{N} \sum_{\substack{h=0\\(h,k)=1}}^{k-1} \int_{\gamma(h,k)} \frac{F(\tau)F(rs\tau)e^{-2\pi i n\tau}}{F(r\tau)F(s\tau)}d\tau. \tag{2.2}$$

Next, for each circle $C_{h,k}$, set $z = -ik^2(\tau - h/k)$, transforming the circle $C_{h,k}$ to the circle K with center $1/2$ and radius $1/2$, and transforming the arc $\gamma(h,k)$ to the arc (not passing through 0) on the latter circle joining the points

$$z_1(h,k) = \frac{k^2 + ikk_1}{k^2 + k_1^2} \qquad \text{and} \qquad z_2(h,k) = \frac{k^2 - ikk_2}{k^2 + k_2^2},$$

where $h_1/k_1 < h/k < h_2/k_2$ are consecutive Farey fractions in $\mathcal{F}_N$. With these changes,

$$p_{r,s}(n) = \sum_{k=1}^{N} \sum_{\substack{h=0\\(h,k)=1}}^{k-1} \int_{z_1(h,k)}^{z_2(h,k)} \frac{F\left(\frac{h}{k} + \frac{iz}{k^2}\right) F\left(rs\left(\frac{h}{k} + \frac{iz}{k^2}\right)\right) e^{-2\pi i n\left(\frac{h}{k} + \frac{iz}{k^2}\right)}\, i}{F\left(r\left(\frac{h}{k} + \frac{iz}{k^2}\right)\right) F\left(s\left(\frac{h}{k} + \frac{iz}{k^2}\right)\right) k^2}dz. \tag{2.3}$$

Next, recall that the Dedekind eta function is defined by

$$\eta(\tau) = e^{\pi i\tau/12}\prod_{k=1}^{\infty}(1 - e^{2\pi i k\tau})$$

and satisfies the transformation formula (see for example Apostol [1], Theorem 3.4)

$$\eta\left(\frac{a\tau + b}{c\tau + d}\right) = \exp\left(\pi i\left(\frac{a+d}{12c} + s(-d,c)\right)\right)\{-i(c\tau + d)\}^{1/2}\eta(\tau) \tag{2.4}$$

whenever $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is an element of the modular group, $c > 0$, and τ lies in the upper half-plane . Thus

$$F(\tau) =$$
$$\exp\left(\frac{\pi i}{12}\left(\tau - \frac{a\tau+b}{c\tau+d} + \frac{a+d}{c}\right)\right)\exp\left(\pi i s(-d,c)\right)\{-i(c\tau+d)\}^{1/2}F\left(\frac{a\tau+b}{c\tau+d}\right).$$

In what follows, for each set of choices for a, c and d, we take b to be $(ad-1)/c$. For $v \in \{1, r, s, rs\}$, $(k, rs) = r_k s_k$ and $\tau = h/k + iz/k^2$, we set $v_k = (v, k)$, so that $(v, v_k) \in \{(1,1), (r, r_k), (s, s_k), (rs, r_k s_k)\}$. We then transform $F(v\tau)$ by setting $c = k/v_k$, $d = -hv/v_k$ and $a = H_{hv/v_k, k/v_k}$, to get

$$F\left(\frac{vh}{k} + \frac{ivz}{k^2}\right) = \exp\left(\frac{\pi v_k^2}{12vz} - \frac{\pi vz}{12k^2}\right)\exp\left(\pi i\, s\left(\frac{hv}{v_k}, \frac{k}{v_k}\right)\right) \times \left\{\frac{vz}{kv_k}\right\}^{1/2} F\left(\frac{H_{hv/v_k, k/v_k}}{k/v_k} + \frac{iv_k^2}{vz}\right). \quad (2.5)$$

On substituting into (2.3), this gives

$$p_{r,s}(n) = \sum_{k=1}^{N}\sum_{\substack{h=0\\(h,k)=1}}^{k-1}\frac{\omega(h,k)\omega(hrs/(r_k s_k), k/(r_k s_k))}{\omega(hr/r_k, k/r_k)\omega(hs/s_k, k/s_k)}\exp\left(\frac{-2\pi i n h}{k}\right)\frac{i}{k^2}$$
$$\times\int_{z_1(h,k)}^{z_2(h,k)}\exp\left(2\pi\left(\frac{r_k s_k \delta_k}{rsz} + \frac{(n-R)z}{k^2}\right)\right)$$
$$\times\frac{F\left(\frac{H_{h,k}}{k} + \frac{i}{z}\right)F\left(\frac{H_{hrs/(r_k s_k), k/(r_k s_k)}}{k/(r_k s_k)} + \frac{ir_k^2 s_k^2}{rsz}\right)}{F\left(\frac{H_{hr/r_k, k/r_k}}{k/r_k} + \frac{ir_k^2}{rz}\right)F\left(\frac{H_{hs/s_k, k/s_k}}{k/s_k} + \frac{is_k^2}{sz}\right)}dz. \quad (2.6)$$

We temporarily fix $v \in \{1, r, s, rs\}$ and introduce the shorthand $g = v_k = (v, k)$. We observe that the congruences

$$H_{vh/g, k/g}(vh/g) \equiv -1 \pmod{k/g} \qquad \text{and} \qquad H_{h,k}h \equiv -1 \pmod{k}$$

imply that

$$vH_{vh/g, k/g} \equiv gH_{h,k} \pmod{k}$$

when $(h, k) = 1$. Since r and s are square-free, we have $(v/g, k) = 1$, and hence the congruence

$$h(v/g)\widetilde{H}_{h,k} \equiv -1 \pmod{k}$$

has a solution $\widetilde{H}_{h,k}$, and we are free to take $H_{h,k} = (v/g)\widetilde{H}_{h,k}$ to be a multiple of v/g. In particular, then, one has $vH_{vh/g,k/g} \equiv gH_{h,k} \pmod{v}$, and since $(v/g, k) = 1$ it follows from (2) and the Chinese Remainder Theorem that

$$vH_{vh/g, k/g} \equiv gH_{h,k} \pmod{vk/g}.$$

Hence the periodicity of $F(\tau)$ implies that

$$F\left(\frac{gH_{vh/g,k/g}}{k}+\frac{ig^2}{vz}\right)=F\left(\frac{g^2H_{h,k}}{vk}+\frac{ig^2}{vz}\right).$$

Put

$$\mu_z=\frac{r_ks_k}{rs}\left(\frac{H_{h,k}}{k}+\frac{i}{z}\right).$$

Then we deduce from (2) that the ratio appearing in (2.6) is

$$\frac{F((rs/r_ks_k)\mu_z)F(r_ks_k\mu_z)}{F((r_ks/s_k)\mu_z)F((s_kr/r_k)\mu_z)}:=G^*(\mu_z), \tag{2.7}$$

where we write

$$G^*(\tau)=\sum_{m=0}^{\infty}c_{m,k}\exp(2\pi im\tau) \tag{2.8}$$

for some coefficients $c_{m,k}$. We note that the coefficients $c_m(h,k)$ occurring in the statement of Theorem 2 satisfy

$$c_m(h,k)=c_{m,k}\exp\left(\frac{2\pi imr_ks_kH_{h,k}}{rsk}\right),$$

so that in particular $|c_m(h,k)|=|c_{m,k}|$. Then (2.6) may be expressed as

$$p_{r,s}(n)=\sum_{\substack{h,k\\(h,k)=1}}\frac{i}{k^2}\Omega_{h,k}e^{-2\pi inh/k} \times\int_{z_1(h,k)}^{z_2(h,k)}\exp\left(2\pi\left(\frac{r_ks_k\delta_k}{rsz}+\frac{(n-R)z}{k^2}\right)\right)G^*(\mu_z)dz,$$

where

$$\Omega_{h,k}=\frac{\omega(h,k)\omega(rsh/(r_ks_k),k/(r_ks_k))}{\omega(rh/r_k,k/r_k)\omega(sh/s_k,k/s_k)}. \tag{2.9}$$

We further introduce the notation

$$\Psi_{m,k}(z)=\exp\left(\frac{2\pi r_ks_k(\delta_k-m)}{rsz}+\frac{2\pi(n-R)z}{k^2}\right), \tag{2.10}$$

which allows us to write

$$p_{r,s}(n)=\sum_{k=1}^{N}\frac{i}{k^2}\sum_{m=0}^{\infty}c_{m,k}\sum_{\substack{0\le h\le k-1\\(h,k)=1}}\Omega_{h,k}\exp\left(\frac{2\pi i}{rsk}(r_ks_kmH_{h,k}-rsnh)\right) \times\int_{z_1(h,k)}^{z_2(h,k)}\Psi_{m,k}(z)\,dz. \tag{2.11}$$

We decompose the sum over m into two parts, $m < \delta_k$ and $m \geq \delta_k$, and write

$$p_{r,s}(n) = P_1(n;N) + P_2(n;N) \tag{2.12}$$

for the resulting decomposition of (2.11). We aim to show that $P_2(n;N)$ contributes a negligible amount to the formula. We find it useful to split the path of integration from $z_1(h,k)$ to $z_2(h,k)$ into the three arcs $[z(k_1), z(N)]$, $[z(-N), z(-k_2)]$, and $[z(-N), z(N)]$, and we further decompose the first two as unions of arcs of the shape $[z(l), z(l+1)]$, where

$$z(l) = \frac{k^2}{k^2+l^2} + \frac{ikl}{k^2+l^2}. \tag{2.13}$$

It is easy to check that each $z(l)$ lies on the circle $|z-1/2| = 1/2$. Since k_1, k, k_2 are denominators of consecutive elements in the Farey sequence of order N, we have $k + k_1 \geq N+1$ and $k + k_2 \geq N+1$, and hence $k_1 \geq N+1-k$ and $-k_2 \leq k-N-1$ for all values of h and k. Moreover, since $hk_1 - h_1 k = h_2 k - hk_2 = 1$, we have $hk_1 \equiv 1 \pmod{k}$ and $hk_2 \equiv -1 \pmod{k}$. It follows that $H_{h,k} \equiv k_2 \equiv -k_1 \pmod{k}$, and hence the condition $-k_2 \leq l \leq k_1 - 1$ is equivalent to a restriction of $H_{h,k}$ to some interval $\mathcal{I}_l$ modulo k. We may therefore interchange the order of summation and integration in (2.11) to obtain

$$P_2(n;N) = \sum_{k=1}^{N} \frac{i}{k^2} \sum_{m=\lceil \delta_k \rceil}^{\infty} c_{m,k}(S_1(m,k) + S_2(m,k) + S_3(m,k)), \tag{2.14}$$

where

$$S_1 = \sum_{l=N+1-k}^{N-1} \int_{z(l)}^{z(l+1)} \Psi_{m,k}(z) \Theta(k,l,m)\, dz,$$

$$S_2 = \sum_{l=-N}^{k-N-2} \int_{z(l)}^{z(l+1)} \Psi_{m,k}(z) \Theta(k,l,m)\, dz,$$

$$S_3 = \int_{z(-N)}^{z(N)} \Psi_{m,k}(z) \Theta(k,l,m)\, dz,$$

and

$$\Theta(k,l,m) = \sum_{\substack{0 \leq h \leq k-1 \\ (h,k)=1 \\ H_{h,k} \in \mathcal{I}_l}} \Omega_{h,k} \exp\left(\frac{2\pi i}{rsk}(r_k s_k m H_{h,k} - rsnh)\right).$$

In order to make further progress, we must develop suitable estimates for $\Theta(k,l,m)$. We take up this task in the next section.

3 Kloosterman sums

In order to estimate S_1, S_2, and S_3, we aim to express $\Theta(k, l, m)$ in terms of Kloosterman sums. As a first step, we are able to remove the restriction on $H_{h,k}$ in the summation at a cost of $O(\log k)$. The argument is similar to that of Hagis [11] (see also Lehner [19]).

Lemma 1 *For each k and m, there exists an integer $j = j(k, m)$ with $0 \leq j \leq k-1$ such that for every l one has*

$$|\Theta(k,l,m)| \ll (1+\log k) \left| \sum_{\substack{h=0 \\ (h,k)=1}}^{k-1} \Omega_{h,k} \exp\left(\frac{2\pi i}{rsk} ((r_k s_k m + rsj) H_{h,k} - rsnh) \right) \right|,$$

where the implicit constant is absolute.

Proof Fix k, l, and m, and let $\mathcal{I}_l = [\alpha, \beta]$, where α and β are integers with $0 \leq \beta - \alpha < k$. By orthogonality, we have

$$\frac{1}{k} \sum_{j=0}^{k-1} \exp\left(\frac{2\pi i j(H-t)}{k} \right) = \begin{cases} 1 & \text{if } H \equiv t \pmod{k} \\ 0 & \text{if } H \not\equiv t \pmod{k} \end{cases},$$

and hence the expression

$$\frac{1}{k} \sum_{t=\alpha}^{\beta} \sum_{j=0}^{k-1} \exp\left(\frac{2\pi i j(H-t)}{k} \right)$$

is 1 if H is congruent mod k to one of the integers in $\mathcal{I}_l$ and 0 otherwise. We therefore have

$$\Theta(k,l,m) = \frac{1}{k} \sum_{j=0}^{k-1} \gamma_j \sum_{\substack{0 \leq h \leq k-1 \\ (h,k)=1}} \Omega_{h,k} \exp\left(\frac{2\pi i}{rsk} ((r_k s_k m + rsj) H_{h,k} - rsnh) \right), \tag{3.1}$$

where

$$\gamma_j = \sum_{t=\alpha}^{\beta} \exp\left(\frac{-2\pi i j t}{k} \right).$$

By summing this geometric progression, we find that

$$|\gamma_j| \leq \min\left(\beta - \alpha, \frac{1}{\sin(\pi j/k)} \right) \leq \min(k, \tfrac{1}{2} ||j/k||^{-1}),$$

where $||\cdot||$ denotes the distance to the nearest integer. One now easily gets (see for example Lemma 3.2 of Baker [2])

$$\sum_{j=0}^{k-1} |\gamma_j| \ll k(1+\log k),$$

and the lemma follows after taking the maximum over j in the inner summation of (3.1).

Write $24rs = AB$, where A is the largest divisor of $24rs$ relatively prime to k, and let $\bar{A}$ denote the multiplicative inverse of A modulo Bk. We note that every prime factor of B is a prime factor of k, whence $\gcd(h,k) = 1$ if and only if $\gcd(h, Bk) = 1$. Moreover, for each such h and k we can find $H_{h,Bk}$ with the property that $hH_{h,Bk} \equiv -1 \pmod{Bk}$ and $A|H_{h,Bk}$. These observations allow us to calculate the $\Omega_{h,k}$ defined by (2.9) rather explicitly.

Lemma 2 *Suppose that* $(h,k) = 1$, *let* A, B, *and* $H_{h,Bk}$ *be as above, and additionally write* $\nu_k = (r_k - 1)(s_k - 1)$, $\sigma_k = (r - r_k)(s - s_k)$, *and*

$$\Phi_{h,k} = \exp\left(\frac{48\pi i \bar{A}}{Bk}(k^2 R - r_k s_k \delta_k) H_{h,Bk}\right),$$

where R *and* δ_k *are as in* (1.2). *When* k *is odd one has*

$$\Omega_{h,k} = \left(\frac{r/r_k}{s_k}\right)\left(\frac{s/s_k}{r_k}\right) \exp\left(\frac{-\pi i k \nu_k}{4 r_k s_k}\right) \exp\left(\frac{-2\pi i}{k r_k s_k}\left(k^2 \delta_k - r_k s_k R\right) h\right) \Phi_{h,k},$$

and when k *is even one has*

$$\Omega_{h,k} = \left(\frac{s_k}{r/r_k}\right)\left(\frac{r_k}{s/s_k}\right) \exp\left(\frac{2\pi i}{k r_k s_k}\left(2k^2 \delta_k + \tfrac{1}{8} k \sigma_k + r_k s_k R\right) h\right) \Phi_{h,k}.$$

Proof When $v \in \{1, r, s, rs\}$, write $g = (v,k)$ and

$$\omega_v(h,k) = \exp\left(\frac{-2\pi i (k^2 - g^2)}{24 k g^3}\left(2hvg + (v^2 h^2 - g^2) H_{vh/g,k/g}\right)\right). \tag{3.2}$$

Then when k is odd, formula (2.4) of Niven [21] gives

$$\omega(vh/g, k/g) = \left(\frac{-hv/g}{k/g}\right) \exp\left(\frac{-\pi i}{4}(k-1)\right) \omega_v(h,k). \tag{3.3}$$

When k is even, the condition that v is square-free implies that hv/g is odd, and hence we may apply formula (2.3) of [21] to obtain

$$\omega(vh/g, k/g) = \left(\frac{-k/g}{hv/g}\right) \exp\left(\frac{-\pi i}{4}\left(2 - \frac{hv}{g^2}(k+g)\right)\right) \omega_v(h,k). \tag{3.4}$$

Since v is square-free, we have $(vh/g, Bk) = 1$. Therefore, as in the argument preceding the statement of the lemma, we may replace each $H_{vh/g,k/g}$ in (3.2) by an integer $H_{hv/g,Bk/g}$ divisible by A, and the argument leading to (2) then gives

$$H_{hv/g,Bk/g} \equiv \frac{g}{v} H_{h,Bk} \pmod{Bk/g},$$

where we recall that $H_{h,Bk}$ is divisible by A and hence by v/g. Substituting into (3.2) now gives

$$\begin{aligned}\omega_v(h,k) &= \exp\left(\frac{-\pi i}{6kg^2}(k^2-g^2)vh\right)\exp\left(\frac{-2\pi i\bar{A}rs}{Bkvg^2}(k^2-g^2)(vh^2-g^2)H_{h,Bk}\right)\\ &= \exp\left(\frac{-\pi i}{12kg^2}(k^2-g^2)vh\right)\exp\left(\frac{2\pi i\bar{A}rs}{Bkv}(k^2-g^2)H_{h,Bk}\right),\end{aligned}$$

upon noting that $v|rs$, $g^2|(k^2-g^2)$, and $h^2H_{h,Bk} \equiv -h \pmod{Bk}$. It now follows with a bit of computation that

$$\frac{\omega_1(h,k)\omega_{rs}(h,k)}{\omega_r(h,k)\omega_s(h,k)} = \exp\left(\frac{-2\pi i}{k}\left(R-\frac{k^2\delta_k}{r_ks_k}\right)hv\right)\Phi_{h,k}.$$

The lemma now follows from (3.3) and (3.4) via routine calculations using the multiplicative properties of the Jacobi symbol.

We now show that the summation on the right hand side of Lemma 1 is a Kloosterman sum with modulus Bk. Fix $j=j(k,m)$ to be the integer in the statement of Lemma 1 for which the expression on the right is maximal and write $T(k,m)$ for the corresponding sum, so that for each l one has

$$\Theta(k,l,m) \ll (1+\log k)T(k,m). \tag{3.5}$$

From the definition of the Dedekind sum (see Section 1), together with (2.9), we see that $\Omega_{h+tk,k}=\Omega_{h,k}$ for all $t\in\mathbb{Z}$. Hence we can write

$$T(k,m)=B^{-1}\sum_{\substack{0\le h\le Bk-1\\(h,Bk)=1}}\Omega_{h,k}\exp\left(\frac{2\pi i}{Bk}\big((24\bar{A}r_ks_km+jB)H_{h,Bk}-Bnh\big)\right),$$

since the definition of B implies that $(h,k)=1$ if and only if $(h,Bk)=1$. Moreover, since Ah runs over a reduced residue system modulo Bk as h does and since $-h^{-1}\equiv AH_{Ah,Bk}\equiv H_{h,Bk} \pmod{Bk}$, we find that

$$T(k,m)=B^{-1}\sum_{\substack{0\le h\le Bk-1\\(h,Bk)=1}}\Omega_{Ah,k}\exp\left(\frac{2\pi i}{Bk}\big((24\bar{A}r_ks_km+jB)\bar{A}H_{h,Bk}-ABnh\big)\right). \tag{3.6}$$

We are now able to express $T(k,m)$ in terms of the Kloosterman sum

$$K(a,b;c)=\sum_{\substack{1\le x\le c\\(x,c)=1}}\exp\left(\frac{2\pi i(ax+b\bar{x})}{c}\right),$$

where $\bar{x}x\equiv 1 \pmod c$. In our case $c=Bk$, and $-H_{h,Bk}$ plays the role of $\bar{x}$. According to Weil's bound (see for example Iwaniec and Kowalski [18], Corollary 11.12) one has

$$K(a,b;c)\ll (a,b,c)^{1/2}c^{1/2+\varepsilon}, \tag{3.7}$$

and this delivers the bound on $\Theta(k,l,m)$ recorded in the following lemma.

Lemma 3 *One has $\Theta(k,l,m) \ll k^{1/2+\varepsilon}$, where the implicit constant depends at most on ε, r, s, and n.*

Proof On substituting the results of Lemma 2 (with h replaced by Ah) into (3.6), we obtain

$$|T(m,k)| = B^{-1}|K(a,b;Bk)|$$

where

$$b = 24\bar{A}^2(r_k s_k(\delta_k - m) - k^2 R) - j\bar{A}B,$$

and

$$a = rs\left(\frac{k\alpha_k}{r_k s_k} + 24R\right) - ABn,$$

where $\alpha_k = -24k^2\delta_k$ if k is odd and $\alpha_k = 48k^2\delta_k + 3k\sigma_k$ if k is even. Since $r_k s_k | k$ and $k | \alpha_k$, any common divisor of a and Bk must also divide the integer

$$u = 24\,(ABn - 24rsR) = 576rs(n-R).$$

In view of the hypothesis that $n > R$, we have $u \neq 0$ and hence $(a,b,Bk) \ll_{r,s,n} 1$. The lemma now follows from (3.5) and (3.7).

4 The error terms

In order to complete the analysis of $P_2(n;N)$, we require an estimate for the growth rate of the coefficients $c_{m,k}$ in (2.11) arising from the expansion (2.8). The following crude bound will suffice for our purposes.

Lemma 4 *One has*

$$c_{m,k} \ll e^{2\pi\sqrt{m}},$$

where the implicit constant is independent of k.

Proof For simplicity, we consider the series

$$g(x) = \sum_{m=0}^{\infty} c_{m,k} x^m,$$

so that (2.8) gives $G^*(\tau) = g(e^{2\pi i\tau})$. Then by (2.7) one has

$$g(x) = \frac{(x^a;x^a)_\infty (x^b;x^b)_\infty}{(x^c;x^c)_\infty (x^d;x^d)_\infty},$$

where $a = r_k s/s_k$, $b = s_k r/r_k$, $c = rs/r_k s_k$, and $d = r_k s_k$. We have

$$\frac{1}{(x^t;x^t)_\infty} = \sum_{l=0}^{\infty} p(l) x^{tl},$$

and it follows that the coefficient of x^m in $[(x^c;x^c)_\infty(x^d;x^d)_\infty]^{-1}$ is bounded above by $(m+1)p(m)^2$. Furthermore, by Euler's Pentagonal number theorem we have

$$(x^t;x^t)_\infty = \sum_{l=-\infty}^{\infty} (-1)^l x^{tl(3l-1)/2},$$

and from this one sees that the coefficient of x^m in $(x^a;x^a)_\infty(x^b;x^b)_\infty$ has absolute value at most $4\sqrt{m}+2$. Hence on applying the well-known Hardy-Ramanujan asymptotic formula [13] for $p(m)$, we deduce that

$$c_{m,k} \ll m^{5/2}p(m)^2 \ll m^{1/2}e^{2\pi(2/3)^{1/2}\sqrt{m}} \ll e^{2\pi\sqrt{m}},$$

and the lemma follows.

We are now able to show that the terms in (2.11) with $m \geq \delta_k$ contribute a negligible amount. First of all, it follows from Lemma 3 and the definitions at the end of Section 2 that

$$S_1 \ll k^{1/2+\varepsilon}\int_{z(N+1-k)}^{z(N)} |\Psi_{m,k}(z)|\,dz, \qquad S_2 \ll k^{1/2+\varepsilon}\int_{z(-N)}^{z(k-N-1)} |\Psi_{m,k}(z)|\,dz,$$

and

$$S_3 \ll k^{1/2+\varepsilon}\int_{z(-N)}^{z(N)} |\Psi_{m,k}(z)|\,dz.$$

Since $\mathrm{Re}(z) \leq 1$ in $|z-1/2| \leq 1/2$, the definition (2.10) immediately gives

$$\Psi_{m,k}(z) \ll_n \exp\biggl(\frac{2\pi r_k s_k}{rsz}(\delta_k - m)\biggr). \tag{4.1}$$

Moreover, one has $\mathrm{Re}(1/z) \geq 1$ in the disk $|z-1/2| \leq 1/2$ and it follows that $\Psi_{m,k}(z) \ll 1$ for all $m \geq \delta_k$. If $\delta_k < 0$ then (4.1) yields $\Psi_{m,k}(z) \ll e^{-2\pi m/(rs)}$, whereas if $\delta_k > 0$ and $m > 2\delta_k$ then we obtain $\Psi_{m,k}(z) \ll e^{-\pi m/(rs)}$.

With the above estimates in hand, it remains to bound the lengths of the various arcs of integration. After recalling (2.13), a simple calculation reveals that

$$|z(N) - z(N-k+1)| \ll k^2/N^2 \quad \text{and} \quad |z(k-N-1) - z(-N)| \ll k^2/N^2,$$

while $|z(N) - z(-N)| \ll 1/N$. Therefore, on shifting the paths of integration from the circle to the respective chords connecting the endpoints, we deduce from (2.14), Lemma 4, and the discussion following (4.1) that

$$\begin{aligned} P_2(n;N) &\ll \sum_{k=1}^{N}(N^{-2}k^{1/2+\varepsilon} + N^{-1}k^{-3/2+\varepsilon})\Biggl(1 + \sum_{m>2\delta_k}|c_{m,k}|e^{-\pi m/(rs)}\Biggr) \\ &\ll N^{-1/2+\varepsilon}. \end{aligned} \tag{4.2}$$

Thus on recalling (2.12) we get

$$p_{r,s}(n) = P_1(n;N) + O(N^{-1/2+\varepsilon}), \tag{4.3}$$

and hence it suffices to analyze $P_1(n;N)$.

5 The main terms

For each k, we now consider the main terms (if any) with $0 \le m < \delta_k$. Recall that K is the circle with center $1/2$ and radius $1/2$, and let $K(-)$ denote this circle traversed in the clockwise direction. We write

$$\int_{z_1(h,k)}^{z_2(h,k)} = \int_{K(-)} - \int_0^{z_1(h,k)} - \int_{z_2(h,k)}^0,$$

and use this to decompose each of the integrals in (2.11). Our aim is to show that the integrals over the arcs $[z_1(h,k), z_2(h,k)]$ in (2.11) can be replaced by integration over $K(-)$, with negligible error. By repeating the argument leading to (2.14), we find that the contribution from $\int_0^{z_1(h,k)}$ and $\int_{z_2(h,k)}^0$ is at most

$$P_3(n;N) = \sum_{k=1}^{N} \frac{1}{k^2} \sum_{0 \le m < \delta_k} (|S_1(k,m)| + |S_2(k,m)| + |S_3(k,m)|). \tag{5.1}$$

Since the coefficient of $1/z$ in the exponent of $\Psi_{m,k}(z)$ is positive when $m < \delta_k$, we keep the path of integration on the circle, where we have $\mathrm{Re}(1/z) = 1$, and hence (4.1) gives $\Psi_{m,k}(z) \ll 1$. Finally, it is easy to show (see for example the proof of Apostol [1], Theorem 5.9) that each of the arcs $[z(N-k+1), z(N)]$, $[z(-N), z(k-N-1)]$, and $[z(-N), z(N)]$ has length $O(k/N)$. It therefore follows from (5.1) and Lemma 3 that

$$P_3(n;N) \ll N^{-1} \sum_{k=1}^{N} k^{-1/2+\varepsilon} \ll N^{-1/2+\varepsilon}. \tag{5.2}$$

On letting $N \to \infty$, we deduce from (4.3) and (5.2) that

$$p_{r,s}(n) = \sum_{k=1}^{\infty} \sum_{m=0}^{\lfloor \delta_k \rfloor} A_{k,m}(n) \frac{i}{k^2} \int_{K(-)} \exp\left(\frac{2\pi r_k s_k}{rsz} (\delta_k - m) + \frac{2\pi z}{k^2} (n - R) \right) dz,$$

where $A_{k,m}(n)$ is as in the statement of Theorem 2. It remains to express the integral over $K(-)$ in terms of modified Bessel functions of the first kind. Setting $w = 1/z$ gives

$$p_{r,s}(n) = \sum_{k=1}^{\infty} \sum_{0 \le m < \delta_k} A_{k,m}(n) \frac{i}{k^2} \times \int_{1-i\infty}^{1+i\infty} \exp\left(\frac{2\pi w r_k s_k}{rs} (\delta_k - m) + \frac{2\pi}{k^2 w} (n - R) \right) \frac{-1}{w^2} dw.$$

We now set

$$t = \frac{2\pi w r_k s_k}{rs} (\delta_k - m) \qquad \text{and} \qquad c = \frac{2\pi r_k s_k}{rs} (\delta_k - m)$$

to get

$$p_{r,s}(n) = \sum_{k=1}^{\infty} \sum_{0 \le m < \delta_k} A_{k,m}(n) \frac{2\pi c}{k^2} \times \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} t^{-2} \exp\left(t + \frac{4\pi^2 r_k s_k}{k^2 rs} (\delta_k - m)(n - R) \frac{1}{t}\right) dt. \quad (5.3)$$

Lastly, we use the formula

$$I_\nu(z) = \frac{(z/2)^\nu}{2\pi i} \int_{c-i\infty}^{c+i\infty} t^{-\nu-1} \exp\left(t + \frac{z^2}{4t}\right) dt$$

(see Watson [29]) with $\nu = 1$ and

$$\frac{z}{2} = \left[\frac{4\pi^2 r_k s_k}{k^2 rs} (\delta_k - m)(n - R)\right]^{1/2}$$

to get, after some simplification,

$$p_{r,s}(n) = \sum_{k=1}^{\infty} \sum_{0 \le m < \delta_k} \frac{2\pi A_{k,m}(n)}{k} \sqrt{\frac{r_k s_k (\delta_k - m)}{rs(n - R)}} \times I_1\left(\frac{4\pi}{k} \sqrt{\frac{r_k s_k}{rs} (\delta_k - m)(n - R)}\right). \quad (5.4)$$

The proof of Theorem 2 is now complete.

6 Convergence behaviour

Obtaining a bound for the error in using the Nth partial sum in Rademacher's series to estimate $p(n)$ is a difficult problem, and bounding the error in using the Nth partial sum of the series at (5.4) to estimate $p_{r,s}(n)$ is likely to be at least as difficult. We do not attempt an analysis of this problem in the present paper. However, we do examine a particular numerical example, to get a feel for the speed and the nature of the convergence.

In the case examined ($r = 14$, $s = 15$, $n = 500$), the convergence of the series is initially very fast, while it seems that once the partial sums of the series get to within 1.0 of the correct value, that convergence then proceeds much more slowly, with (for $k \ge rs$) the greatest contributions to the sum of the series coming from those terms with $k \equiv 0 \pmod{rs}$, and with the contributions from the terms for the other k being negligible in comparison. However, it is possible that the convergence behaviour may be different, if r and s have a different number of prime factors than in the example.

As an illustration of the convergence behaviour, we consider the convergence of the sum of the series to

$$p_{14,15}(500) = 310,093,947,025,073,675,623,$$

Table 1

N	S_N	$p_{14,15}(500) - S_N$
1	310093947025049932429.8505	-2.374319315×10^7
2	310093947025073675628.9283	5.9283
3	310093947025073675414.3591	−208.6409
4	310093947025073675623.3258	0.3258
5	310093947025073675623.3258	0.3258
6	310093947025073675623.3723	0.3723
7	310093947025073675623.3723	0.3723
8	310093947025073675623.3723	0.3723
9	310093947025073675623.2793	0.2793
10	310093947025073675623.2793	0.2793
11	310093947025073675623.4447	0.4447

by examining the difference $p_{14,15}(500) - S_N$, where S_N is the Nth partial sum of the series. We tabulate the values for $1 \leq N \leq 11$ in Table 6 to show the very fast convergence initially.

Note that the terms in the series corresponding to $k = 5$, $k = 7$ and $k = 10$ are zero, since $\delta_5, \delta_7, \delta_{10} < 0$, so that each of the inner sums over m are empty, and thus contribute zero to the value of the series (the term in the series corresponding to $k = 8$ is also zero, but this is because the terms in the inner sum over m add to zero).

We next plot (Figure 1) the values for $1 \leq N \leq 750$ (the large initial values lie outside the range of the plot) to show how the terms in the sum corresponding to $k = 210, 420, 630, \ldots$ (multiples of $r \times s = 14 \times 15$) contribute much more to the value of tail of the series than values of $k \not\equiv 0 \pmod{rs}$.

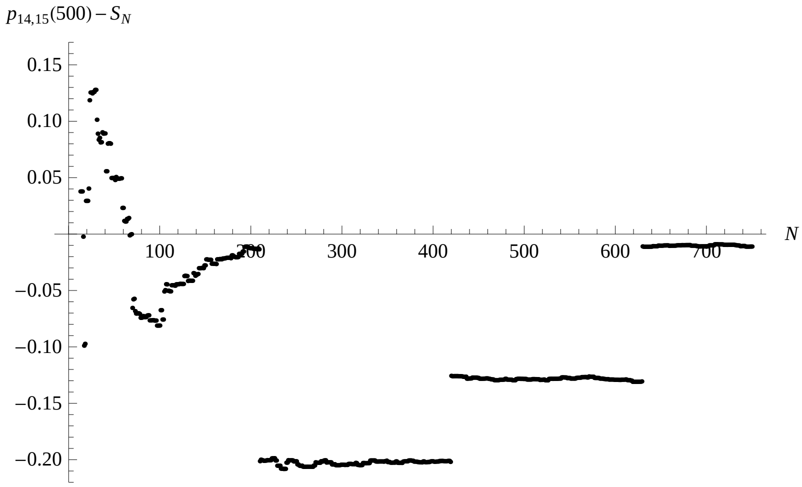

Fig. 1 The convergence of the series to $p_{14,15}(500)$. The plot shows the values of $p_{14,15}(500) - S_N$, where S_N is the Nth partial sum of the series. Note the jumps in the values of the partial sums for $k = 210, 420$ and 630.

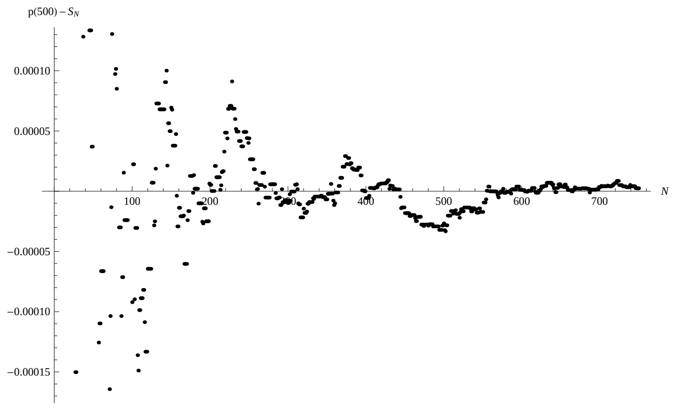

Fig. 2 The convergence of Rademacher's series to $p(500)$. Note the more erratic convergence behaviour, compared that of the series for $p_{14,15}(500)$.

We remark that this apparent step-like convergence behaviour of the series for $p_{r,s}(n)$ is in contrast to the apparent convergence behaviour of Rademacher's series for $p(n)$, which is more erratic. Figure 2 is a plot of the difference $p(500) - S_N$, $1 \leq N \leq 750$, where S_N is the Nth partial sum of Rademacher's series.

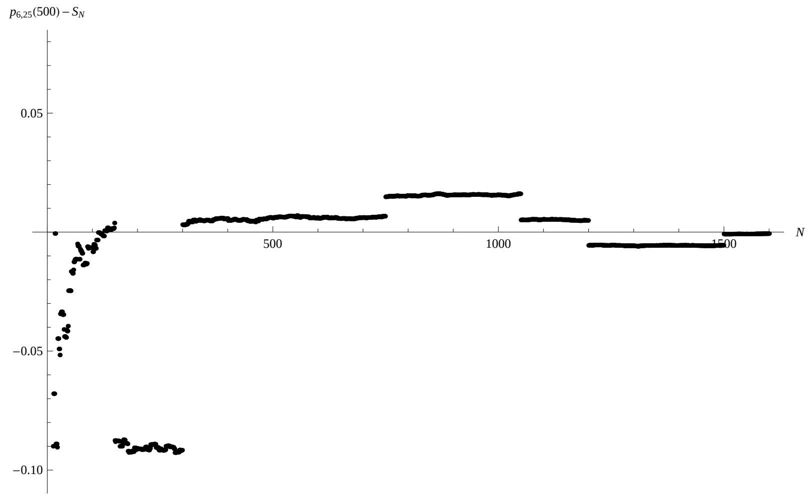

Fig. 3 The series corresponding to $p_{6,25}(500)$ appears to converge to $p_{6,25}(500)$, despite the fact that 25 is a square.

We conclude by remarking that experimental evidence suggests that the requirement that r and s be square-free may be dropped, although it is not possible to employ the arguments used to get the Kloosterman sum estimates in this case. For example, seven terms of the series for

$$p_{6,25}(500) = 42,305,606,435,448,427,065$$

appear to be sufficient to get within 0.5 of $p_{6,25}(500)$. Figure 3 is a plot of difference $p_{6,25}(500) - S_N$, $1 \leq N \leq 1600$, where once again S_N is the Nth partial sum of the series in Theorem 2.

Note that the convergence of the series for $p_{6,25}(500)$ exhibits the same step-like behaviour seen above in the convergence of the series for $p_{14,15}(500)$, with the steps this time being multiples of $150 = 6 \times 25$.

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