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# SOME APPLICATIONS OF A BAILEY-TYPE TRANSFORMATION

JAMES MC LAUGHLIN AND PETER ZIMMER

ABSTRACT. If  $k$  is set equal to  $aq$  in the definition of a WP Bailey pair,

$$\beta_n(a, k) = \sum_{j=0}^n \frac{(k/a)_{n-j} (k)_{n+j}}{(q)_{n-j} (aq)_{n+j}} \alpha_j(a, k),$$

this equation reduces to  $\beta_n = \sum_{j=0}^n \alpha_j$ .

This seemingly trivial relation connecting the  $\alpha_n$ 's with the  $\beta_n$ 's has some interesting consequences, including several basic hypergeometric summation formulae, a connection to the Prouhet-Tarry-Escott problem, some new identities of the Rogers-Ramanujan-Slater type, some new expressions for false theta series as basic hypergeometric series, and new transformation formulae for poly-basic hypergeometric series.

## 1. INTRODUCTION

We begin by recalling a construction of Andrews [1]. If a pair of sequences  $(\alpha_n(a, k), \beta_n(a, k))$  satisfy

$$(1.1) \quad \beta_n(a, k) = \sum_{j=0}^n \frac{(k/a)_{n-j} (k)_{n+j}}{(q)_{n-j} (aq)_{n+j}} \alpha_j(a, k),$$

then so does the pair  $(\alpha'_n(a, k), \beta'_n(a, k))$ , where

$$(1.2) \quad \begin{aligned} \alpha'_n(a, k) &= \frac{(\rho_1, \rho_2)_n}{(aq/\rho_1, aq/\rho_2)_n} \left(\frac{k}{c}\right)^n \alpha_n(a, c), \\ \beta'_n(a, k) &= \frac{(k\rho_1/a, k\rho_2/a)_n}{(aq/\rho_1, aq/\rho_2)_n} \\ &\quad \times \sum_{j=0}^n \frac{(1 - cq^{2j})(\rho_1, \rho_2)_j (k/c)_{n-j} (k)_{n+j}}{(1 - c)(k\rho_1/a, k\rho_2/a)_n (q)_{n-j} (qc)_{n+j}} \left(\frac{k}{c}\right)^j \beta_j(a, c), \end{aligned}$$

with  $c = k\rho_1\rho_2/aq$ . A pair of sequences satisfying (1.1) is termed a *WP-Bailey pair*. If  $k = 0$ , the pair of sequences become what is termed a *Bailey pair relative to a*.

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Bailey [4, 5] used the  $q$ -Gauss sum,

$$(1.3) \quad {}_2\phi_1(a, b; c; q, c/ab) = \frac{(c/a, c/b; q)_\infty}{(c, c/ab; q)_\infty},$$

to get that, if  $(\alpha_n, \beta_n)$  are a Bailey pair relative to  $a$ , then

$$(1.4) \quad \sum_{n=0}^{\infty} (y, z; q)_n \left(\frac{aq}{yz}\right)^n \beta_n = \frac{(aq/y, aq/z; q)_\infty}{(aq, aq/yz; q)_\infty} \sum_{n=0}^{\infty} \frac{(y, z; q)_n}{(aq/y, aq/z; q)_n} \left(\frac{x}{yz}\right)^n \alpha_n.$$

Slater, in [21] and [22], subsequently used this transformation of Bailey to derive 130 identities of the Rogers-Ramanujan type.

The first major variations in Bailey's construct at (1.4) appear to be due to Bressoud [7]. Another variation was given by Singh in [20]. All of these variations were put in a more formal setting by Andrews in [1], where he introduced the generalization of the standard Bailey pair defined above.

In the same paper Andrews also described a second way to construct a new WP-Bailey pair from a given pair. These two constructions allowed a "tree" of WP-Bailey pairs to be generated from a single WP-Bailey pair. These two branches of the WP-Bailey tree were further investigated by Andrews and Berkovich in [2]. Spiridonov [23] derived an elliptic generalization of Andrews first WP-Bailey chain, and Warnaar [25] added four new branches to the WP-Bailey tree, two of which had generalizations to the elliptic level. More recently, and motivated in part by the papers above, Liu and Ma [14] introduced the idea of a general WP-Bailey chain (as a solution to a system of linear equations), and added one new branch to the WP-Bailey tree.

As we might expect, Andrews generalization of a Bailey pair leads to a generalization of (1.4). Indeed Andrews WP-Bailey chain at (1.2) can easily be shown to imply the following result (substitute the expression for  $\alpha'_n(a, k)$  in (1.1), set the two expressions for  $\beta'_n(a, k)$  equal, and let  $n \rightarrow \infty$ ). Note that setting  $k = 0$  recovers Bailey's transformation at (1.4). (We initially derived (1.6) in a way similar to Bailey's derivation of (1.4), before realizing that it followed from Andrews' construction (1.2).)

**Theorem 1.** *Under suitable convergence conditions, if  $(\alpha_n(a, k), \beta_n(a, k))$  satisfy*

$$(1.5) \quad \beta_n(a, k) = \sum_{j=0}^n \frac{(k/a)_{n-j} (k)_{n+j}}{(q)_{n-j} (aq)_{n+j}} \alpha_j(a, k),$$

then

$$(1.6) \quad \sum_{n=0}^{\infty} \frac{(1 - kq^{2n})(\rho_1, \rho_2; q)_n}{(1 - k)(kq/\rho_1, kq/\rho_2; q)_n} \left(\frac{aq}{\rho_1\rho_2}\right)^n \beta_n(a, k) = \frac{(kq, kq/\rho_1\rho_2, aq/\rho_1, aq/\rho_2; q)_\infty}{(kq/\rho_1, kq/\rho_2, aq/\rho_1\rho_2, aq; q)_\infty} \sum_{n=0}^{\infty} \frac{(\rho_1, \rho_2; q)_n}{(aq/\rho_1, aq/\rho_2; q)_n} \left(\frac{aq}{\rho_1\rho_2}\right)^n \alpha_n(a, k).$$

In the present paper we investigate what at first glance may appear to be a trivial special case of Theorem 1.

**Corollary 1.** *If  $\beta_n = \sum_{r=0}^n \alpha_r$ , then assuming both series converge,*

$$(1.7) \quad \sum_{n=0}^{\infty} \frac{(q\sqrt{xyz}, -q\sqrt{xyz}, y, z; q)_n x^n \beta_n}{(\sqrt{xyz}, -\sqrt{xyz}, qxy, qxz; q)_n} = \frac{(1-xy)(1-xz)}{(1-x)(1-xyz)} \sum_{n=0}^{\infty} \frac{(y, z; q)_n x^n \alpha_n}{(xy, xz; q)_n}.$$

*Proof.* Let  $k = xyz$ ,  $a = xyz/q$ ,  $\rho_1 = y$  and  $\rho_2 = z$  in Theorem 1.  $\square$

This seemingly trivial relation connecting the  $\alpha_n$ 's with the  $\beta_n$ 's has some interesting consequences, including several basic hypergeometric summation formulae, a connection to the Prouhet-Tarry-Escott problem, some new identities of the Rogers-Ramanujan-Slater type, some new expressions for false theta series as basic hypergeometric series, and new transformation formulae for poly-basic hypergeometric series.

We employ the usual notations. Let  $a$  and  $q$  be complex numbers, with  $|q| < 1$  unless otherwise stated. Then

$$(a)_0 = (a; q)_0 := 1, \quad (a)_n = (a; q)_n := \prod_{j=0}^{n-1} (1 - aq^j), \quad \text{for } n \in \mathbb{N},$$

$$(a_1; q)_n (a_2; q)_n \cdots (a_k; q)_n = (a_1, a_2, \dots, a_k; q)_n,$$

$$(a; q)_\infty := \prod_{j=0}^{\infty} (1 - aq^j),$$

$$(a_1; q)_\infty (a_2; q)_\infty \cdots (a_k; q)_\infty = (a_1, a_2, \dots, a_k; q)_\infty.$$

An  ${}_r\phi_s$  basic hypergeometric series is defined by

$${}_r\phi_s \left[ \begin{matrix} a_1, a_2, \dots, a_r \\ b_1, \dots, b_s \end{matrix}; q, x \right] = \sum_{n=0}^{\infty} \frac{(a_1; q)_n (a_2; q)_n \cdots (a_r; q)_n}{(q; q)_n (b_1; q)_n \cdots (b_s; q)_n} \left( (-1)^n q^{n(n-1)/2} \right)^{s+1-r} x^n.$$

For future use we also recall the  $q$ -binomial theorem,

$$(1.8) \quad \sum_{n=0}^{\infty} \frac{(a; q)_n}{(q; q)_n} z^n = \frac{(az; q)_\infty}{(z; q)_\infty}.$$

## 2. VARIOUS SUMMATION FORMULAE FOR BASIC HYPERGEOMETRIC SERIES

We next derive a number of transformation formulae for basic hypergeometric series, transformations that give rise to summation formulae for particular choices of the parameters.

**Corollary 2.** For  $q$  and  $x$  inside the unit disc,

$$(2.1) \quad \sum_{n=0}^{\infty} \frac{(q\sqrt{xyz}, -q\sqrt{xyz}, y, z; q)_{2n} x^{2n}}{(\sqrt{xyz}, -\sqrt{xyz}, qxy, qxz; q)_{2n}} = \frac{(1-xy)(1-xz)}{(1-x)(1-xyz)} \sum_{n=0}^{\infty} \frac{(y, z; q)_n (-x)^n}{(xy, xz; q)_n}.$$

$$(2.2) \quad \sum_{n=0}^{\infty} \frac{(1 - q^{2n+1}/x)(q/x^2; q)_{2n} x^{2n}}{(q; q)_{2n+1}} = \frac{1}{1+x} \frac{(q/x; q)_{\infty}}{(x; q)_{\infty}}, \quad x \neq 0.$$

*Proof.* In Corollary 1 let  $\alpha_r = (-1)^r$  to get (2.1). Then set  $y = q/x$ ,  $z = q/x^2$ , apply (1.8) to the right side, replace  $x$  by  $-x$  and (2.2) follows.  $\square$

**Corollary 3.** For  $q$  and  $x$  inside the unit disc,

$$(2.3) \quad \sum_{n=0}^{\infty} \frac{(q\sqrt{xyz}, -q\sqrt{xyz}, y, z; q)_n x^n (n+1)}{(\sqrt{xyz}, -\sqrt{xyz}, qxy, qxz; q)_n} = \frac{(1-xy)(1-xz)}{(1-x)(1-xyz)} \sum_{n=0}^{\infty} \frac{(y, z; q)_n x^n}{(xy, xz; q)_n}.$$

$$(2.4) \quad \sum_{n=0}^{\infty} \frac{(1 + q^{n+1}/x)(q/x^2; q)_n x^n (n+1)}{(q; q)_{n+1}} = \frac{1}{1-x} \frac{(q/x; q)_{\infty}}{(x; q)_{\infty}}.$$

*Proof.* Set  $\alpha_n = 1$  in Corollary 1 to get (2.3). The identity at (2.4) follows from (2.3) upon setting  $y = q/x^2$ ,  $z = q/x$ , using (1.8) to sum the right side and then simplifying.  $\square$

**Corollary 4.** For  $q$ ,  $x$  and  $u$  all inside the unit disc,

$$(2.5) \quad \sum_{n=0}^{\infty} \frac{(1 + q^{n+1}/x)(q/x^2; q)_n x^n (1 - u^{n+1})}{(q; q)_{n+1}} = \frac{1-u}{1-x} \frac{(qu/x; q)_{\infty}}{(xu; q)_{\infty}}.$$

*Proof.* Set  $\alpha_n = u^n$ ,  $y = q/x$  and  $z = q/x^2$ . Now apply the  $q$ -binomial theorem (1.8) to the right side.  $\square$

**Corollary 5.**

$$(2.6) \quad {}_5\phi_4 \left[ \begin{matrix} q\sqrt{xyz}, -q\sqrt{xyz}, y, z, cq \\ \sqrt{xyz}, -\sqrt{xyz}, qxy, qxz \end{matrix}; q, x \right] = \frac{(1-xy)(1-xz)}{(1-x)(1-xyz)} {}_3\phi_2 \left[ \begin{matrix} y, z, c \\ xy, xz \end{matrix}; q, xq \right].$$

$$(2.7) \quad {}_3\phi_2 \left[ \begin{matrix} -qxy, y, x \\ -xy, qx^2y \end{matrix}; q, x \right] = \frac{1}{1+xy} \frac{(x^2, qxy; q)_{\infty}}{(qx^2y, x; q)_{\infty}}.$$

*Proof.* We define  $\alpha_0 = 1$ , and for  $n > 0$ ,

$$\alpha_n = \frac{(cq; q)_n}{(q; q)_n} - \frac{(cq; q)_{n-1}}{(q; q)_{n-1}} = \frac{(c; q)_n}{(q; q)_n} q^n.$$

Substitution into (1.7) immediately gives (2.6). Equation (2.7) follows upon letting  $c = x/q$ ,  $z = xy$  and using (1.3) to sum the resulting right side and simplifying.  $\square$

### 3. TRANSFORMATION FORMULAE FOR BASIC- AND POLYBASIC HYPERGEOMETRIC SERIES

In contrast to the situation with basic hypergeometric series, most (possibly all) summation formulae for poly-basic hypergeometric series arise because the series involved telescope. This means that the terms in such an identity may be inserted in (1.7) to produce a transformation formula for polybasic hypergeometric series containing an additional base. Setting all the bases equal to  $q^m$ , for some integer  $m$ , then gives a transformation formula for basic hypergeometric series. We give one example in the next corollary, which contains a transformation formula connecting polybasic hypergeometric series with five independent bases.

**Corollary 6.** *Let  $P, p, Q, q, R$  and  $x$  all lie inside the unit disc, and let  $a, b, c, y$  and  $z$  be complex numbers such that the denominators below are bounded away from zero. Then*

$$(3.1) \quad \sum_{n=0}^{\infty} \frac{(q\sqrt{xyz}, -q\sqrt{xyz}, y, z; q)_n}{(\sqrt{xyz}, -\sqrt{xyz}, qxy, qxz; q)_n} \frac{(ap^2; p^2)_n (bP^2; P^2)_n}{\left(\frac{PQR}{p}, \frac{PQR}{p}\right)_n \left(\frac{apPQ}{cR}, \frac{pPQ}{R}\right)_n} \\ \times \frac{(cR^2; R^2)_n \left(\frac{aQ^2}{bc}; Q^2\right)_n}{\left(\frac{apQR}{bP}, \frac{pQR}{P}\right)_n \left(\frac{bcPQR}{Q}, \frac{pPR}{Q}\right)_n} x^n \\ = \frac{(1-xy)(1-xz)}{(1-x)(1-xyz)} \times \\ \sum_{n=0}^{\infty} \frac{(y, z; q)_n}{(xy, xz; q)_n} \frac{(1-ap^n P^n Q^n R^n) \left(1-b\frac{p^n P^n}{Q^n R^n}\right) \left(1-\frac{P^n Q^n}{cp^n R^n}\right) \left(1-\frac{ap^n Q^n}{bcP^n R^n}\right)}{(1-a)(1-b) \left(1-\frac{1}{c}\right) \left(1-\frac{a}{bc}\right)} \\ \times \frac{(a; p^2)_n (b; P^2)_n}{\left(\frac{PQR}{p}, \frac{PQR}{p}\right)_n \left(\frac{apPQ}{cR}, \frac{pPQ}{R}\right)_n} \frac{(c; R^2)_n \left(\frac{a}{bc}; Q^2\right)_n}{\left(\frac{apQR}{bP}, \frac{pQR}{P}\right)_n \left(\frac{bcPQR}{Q}, \frac{pPR}{Q}\right)_n} (xR^2)^n; \\ (3.2) \quad \sum_{n=0}^{\infty} \frac{(q\sqrt{xyz}, -q\sqrt{xyz}, y, z; q)_n}{(\sqrt{xyz}, -\sqrt{xyz}, qxy, qxz; q)_n} \frac{(aq^m, bq^m, cq^m, \frac{aq^m}{bc}; q^m)_n}{\left(\frac{a}{c}q^m, \frac{a}{b}q^m, bcq^m, q^m; q^m\right)_n} x^n \\ = \frac{(1-xy)(1-xz)}{(1-x)(1-xyz)} \times$$

$$\sum_{n=0}^{\infty} \frac{(y, z; q)_n}{(xy, xz; q)_n} \frac{(q^m \sqrt{a}, -q^m \sqrt{a}, a, b, c, \frac{a}{bc}; q^m)_n}{(\sqrt{a}, -\sqrt{a}, \frac{a}{c} q^m, \frac{a}{b} q^m, bcq^m, q^m; q^m)_n} (xq^m)^n.$$

*Proof.* We use the special case  $m = 0$ ,  $d = 1$  of the identity of Subbarao and Verma labeled (2.2) in [24], namely,

$$(3.3) \quad \sum_{k=0}^n \frac{(1 - ap^k P^k Q^k R^k) \left(1 - b \frac{p^k P^k}{Q^k R^k}\right) \left(1 - \frac{P^k Q^k}{cp^k R^k}\right) \left(1 - \frac{ap^k Q^k}{bcP^k R^k}\right)}{(1-a)(1-b) \left(1 - \frac{1}{c}\right) \left(1 - \frac{a}{bc}\right)} \\ \times \frac{(a; p^2)_k (b; P^2)_k (c; R^2)_k \left(\frac{a}{bc}; Q^2\right)_k}{\left(\frac{PQR}{p}, \frac{PQR}{p}\right)_k \left(\frac{apPQ}{cR}, \frac{pPQ}{R}\right)_k \left(\frac{apQR}{bP}, \frac{pQR}{P}\right)_k \left(\frac{bcP^2R}{Q}, \frac{pPR}{Q}\right)_k} R^{2k} \\ = \frac{(ap^2; p^2)_n (bP^2; P^2)_n (cR^2; R^2)_n \left(\frac{aQ^2}{bc}; Q^2\right)_n}{\left(\frac{PQR}{p}, \frac{PQR}{p}\right)_n \left(\frac{apPQ}{cR}, \frac{pPQ}{R}\right)_n \left(\frac{apQR}{bP}, \frac{pQR}{P}\right)_n \left(\frac{bcP^2R}{Q}, \frac{pPR}{Q}\right)_n},$$

and then in (1.7) let  $\alpha_i$  be the  $i$ -th term in the sum above, and let  $\beta_n$  be the quantity on the right side above.

The identity at (3.2) follows upon setting  $P = Q = p = R = q^{m/2}$  and simplifying.  $\square$

#### 4. A CONNECTION WITH THE PROUHET-TARRY-ESCOTT PROBLEM

We begin with a simple example.

##### Corollary 7.

$$(4.1) \quad {}_6\phi_5 \left[ \begin{matrix} q\sqrt{xyz}, -q\sqrt{xyz}, y, z, aq, bq \\ \sqrt{xyz}, -\sqrt{xyz}, qxy, qxz, abq \end{matrix}; q, x \right] \\ = \frac{(1-xy)(1-xz)}{(1-x)(1-xyz)} {}_4\phi_3 \left[ \begin{matrix} y, z, a, b \\ xy, xz, abq \end{matrix}; q, xq \right].$$

*Proof.* This time, in Corollary 1, define  $\alpha_0 = 1$ , and for  $n > 0$ ,

$$\alpha_n = \frac{(aq, bq; q)_n}{(abq, q; q)_n} - \frac{(aq, bq; q)_{n-1}}{(abq, q; q)_{n-1}} = \frac{(a, b; q)_n q^n}{(abq, q; q)_n}.$$

The result follows as above.  $\square$

The telescoping approach used in Corollary 7 can be generalized in one direction. We have the following result.

**Proposition 1.** *Let  $x, y$  and  $q$  be complex numbers with  $|x|, |q| < 1$ . Suppose  $a_1, a_2, \dots, a_m$  are non-zero complex numbers and let  $b_1, b_2, \dots, b_{m-1}$  satisfy*

$$(4.2) \quad (z-1) \prod_{i=1}^{m-1} (z-b_i) = \prod_{i=1}^m (z-a_i) - \prod_{i=1}^m (1-a_i).$$

Suppose further that  $b_i \neq 0$ , for  $1 \leq i \leq m-1$ . Then

$$(4.3) \quad m+4\phi_{m+3} \left[ \begin{matrix} q\sqrt{xyz}, -q\sqrt{xyz}, y, z, a_1q, \dots, a_{m-1}q, a_mq; q, x \\ \sqrt{xyz}, -\sqrt{xyz}, qxy, qxz, b_1q, \dots, b_{m-1}q \end{matrix} \right] \\ = \frac{(1-xy)(1-xz)}{(1-x)(1-xyz)} m+2\phi_{m+1} \left[ \begin{matrix} y, z, a_1, \dots, a_{m-1}, a_m; q, xq^m \\ xy, xz, b_1q, \dots, b_{m-1}q \end{matrix} \right].$$

*Proof.* Define  $\alpha_0 = 1$ , and for  $n \geq 1$ , set

$$\alpha_n = \frac{(a_1q, a_2q, \dots, a_{m-1}q, a_mq; q)_n}{(b_1q, b_2q, \dots, b_{m-1}q, q; q)_n} - \frac{(a_1q, a_2q, \dots, a_{m-1}q, a_mq; q)_{n-1}}{(b_1q, b_2q, \dots, b_{m-1}q, q; q)_{n-1}}.$$

By (4.2),

$$\alpha_n = \frac{(a_1, a_2, \dots, a_{m-1}, a_m; q)_n}{(b_1q, b_2q, \dots, b_{m-1}q, q; q)_n} q^{mn}$$

and clearly

$$(4.4) \quad \beta_n = \sum_{r=0}^n \alpha_r = \frac{(a_1q, a_2q, \dots, a_{m-1}q, a_mq; q)_n}{(b_1q, b_2q, \dots, b_{m-1}q, q; q)_n}.$$

The result follows from Corollary 1.  $\square$

The fundamental theorem of algebra guarantees that there is no shortage of sets of complex numbers  $a_1, a_2, \dots, a_m, b_1, b_2, \dots, b_{m-1}$  satisfying (4.2), but for  $m > 4$  it is still a problem to find explicit examples. However, a related problem in number theory provides solutions for  $m \leq 10$  and  $m = 12$ .

The *Prouhet-Tarry-Escott* problem asks for two distinct multisets of integers  $A = \{a_1, \dots, a_m\}$  and  $B = \{b_1, \dots, b_m\}$  such that

$$(4.5) \quad \sum_{i=1}^m a_i^e = \sum_{i=1}^m b_i^e, \text{ for } e = 1, 2, \dots, k,$$

for some integer  $k < m$ . If  $k = m-1$ , such a solution is called *ideal*. We write

$$(4.6) \quad \{a_1, \dots, a_m\} \stackrel{k}{=} \{b_1, \dots, b_m\}$$

to denote a solution to the Prouhet-Tarry-Escott problem.

The connection between the Prouhet-Tarry-Escott problem and the problem mentioned above is contained in the following proposition (see [6], page 2065).

**Proposition 2.** *The multisets  $A = \{a_1, \dots, a_m\}$  and  $B = \{b_1, \dots, b_m\}$  form an ideal solution to the Prouhet-Tarry-Escott problem if and only if*

$$\prod_{i=1}^m (z - a_i) - \prod_{i=1}^m (z - b_i) = C,$$

for some constant  $C$ .



Note that the fact that  $b_m = 1$  is not a problem, since if

$$\{a_1, \dots, a_m\} \stackrel{m-1}{=} \{b_1, \dots, b_m\},$$

then

$$\{Ma_1 + K, \dots, Ma_m + K\} \stackrel{m-1}{=} \{Mb_1 + K, \dots, Mb_m + K\},$$

for constants  $M$  and  $K$  (see Lemma 1 in [8], for example).

Parametric ideal solutions are known for  $m = 1, \dots, 8$  and particular numerical solutions are known for  $m = 9, 10$  and  $12$ . Although every ideal solution to the Prouhet-Tarry-Escott problem gives rise to a transformation between basic hypergeometric series, we will consider just one example. Note also that it is not necessary, for our purposes, that the  $a_i$ 's and  $b_i$ 's be integers. As above, we assume  $x$ ,  $y$  and  $q$  are complex numbers, with  $|x|, |q| < 1$ .

**Corollary 8.** *Let  $m$  and  $n$  be non-zero complex numbers. Set*

$$(4.7) \quad \begin{aligned} a_1 &= -3m^2 + 7nm - 2n^2 + 1, & b_1 &= -3m^2 + 8nm + n^2 + 1, \\ a_2 &= -2m^2 + 8nm + 2n^2 + 1, & b_2 &= -2m^2 + 3nm - 3n^2 + 1, \\ a_3 &= -m^2 - n^2 + 1, & b_3 &= -m^2 + 10nm - n^2 + 1, \\ a_4 &= 2m^2 + 3nm + n^2 + 1, & b_4 &= 2m^2 + 2nm - 2n^2 + 1, \\ a_5 &= m^2 + 2nm - 3n^2 + 1, & b_5 &= m^2 + 7nm + 2n^2 + 1, \\ a_6 &= 10mn + 1. \end{aligned}$$

Then

$$(4.8) \quad {}_{10}\phi_9 \left[ \begin{matrix} q\sqrt{xyz}, -q\sqrt{xyz}, y, z, a_1q, a_2q, a_3q, a_4q, a_5q, a_6q \\ \sqrt{xyz}, -\sqrt{xyz}, qxy, qxz, b_1q, b_2q, b_3q, b_4q, b_5q \end{matrix}; q, x \right] \\ = \frac{(1-xy)(1-xz)}{(1-x)(1-xyz)} {}_8\phi_7 \left[ \begin{matrix} y, z, a_1, a_2, a_3, a_4, a_5, a_6 \\ xy, xz, b_1q, b_2q, b_3q, b_4q, b_5q \end{matrix}; q, xq^6 \right].$$

*Proof.* We have from page 629–30 and Lemma 1 in [8], that if

$$(4.9) \quad \begin{aligned} a_1 &= -5m^2 + 4nm - 3n^2 + K, & b_1 &= -5m^2 + 6nm + 3n^2 + K, \\ a_2 &= -3m^2 + 6nm + 5n^2 + K, & b_2 &= -3m^2 - 4nm - 5n^2 + K, \\ a_3 &= -m^2 - 10nm - n^2 + K, & b_3 &= -m^2 + 10nm - n^2 + K, \\ a_4 &= 5m^2 - 4nm + 3n^2 + K, & b_4 &= 5m^2 - 6nm - 3n^2 + K, \\ a_5 &= 3m^2 - 6nm - 5n^2 + K, & b_5 &= 3m^2 + 4nm + 5n^2 + K, \\ a_6 &= m^2 + 10nm + n^2 + K, & b_6 &= m^2 - 10nm + n^2 + K, \end{aligned}$$

then

$$\{a_1, a_2, a_3, a_4, a_5, a_6\} \stackrel{5}{=} \{b_1, b_2, b_3, b_4, b_5, b_6\}.$$

We set  $b_6 = 1$ , solve for  $K$  and back-substitute in (4.9). We then replace  $m$  by  $m/\sqrt{2}$  and  $n$  by  $n/\sqrt{2}$ . This leads to the values for the  $a_i$ 's and  $b_i$ 's given at (4.7) and the result follows, as before, from Proposition 1.  $\square$

We also note each ideal solution to the Prouhet-Tarry-Escott problem leads to an infinite summation formula, upon letting  $n \rightarrow \infty$  in (4.4). We give one example.

**Corollary 9.** *Let  $m$  be a non-zero complex number. Set*

$$\{a_i\}_{i=1}^{12} = \{1 + 170m, 1 + 126m, 1 + 209m, 1 + 87m, 1 + 234m, 1 + 62m, \\ 1 + 275m, 1 + 21m, 1 + 288m, 1 + 8m, 1 + 299m, 1 - 3m\},$$

$$\{b_i\}_{i=1}^{11} = \{1 + 183m, 1 + 113m, 1 + 195m, 1 + 101m, 1 + 242m, 1 + 54m, \\ 1 + 269m, 1 + 27m, 1 + 294m, 1 + 2m, 1 + 296m\}.$$

Then

$$(4.10) \quad {}_{12}\phi_{11} \left[ \begin{matrix} a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10}, a_{11}, a_{12} \\ b_1q, b_2q, b_3q, b_4q, b_5q, b_6q, b_7q, b_9q, b_{10}q, b_{11}q \end{matrix}; q, q^{12} \right] \\ = \frac{(a_1q, a_2q, a_3q, a_4q, a_5q, a_6q, a_7q, a_8q, a_9q, a_{10}q, a_{11}q, a_{12}q; q)_{\infty}}{(b_1q, b_2q, b_3q, b_4q, b_5q, b_6q, b_7q, b_9q, b_{10}q, b_{11}q, q; q)_{\infty}}$$

*Proof.* We use a result of Nuutti Kuosa, Jean-Charles Meyrignac and Chen Shuwen (see [19]), namely, that if

(4.11)

$$A = \{K + 22m, K - 22m, K + 61m, K - 61m, K + 86m, K - 86m, \\ K + 127m, K - 127m, K + 140m, K - 140m, K + 151m, K - 151m\}, \\ B = \{K + 35m, K - 35m, K + 47m, K - 47m, K + 94m, K - 94m, \\ K + 121m, K - 121m, K + 146m, K - 146m, K + 148m, K - 148m\},$$

then

$$A \stackrel{11}{=} B.$$

□

Remark: Note that while the  $K$  and  $m$  are irrelevant in (4.11) in so far as finding integer solutions to the Prouhet-Tarry-Escott problem (since the solution derived another solution by scaling by  $m$  and translating by  $K$  is trivially equivalent to the original solution), solving  $B_{12} = 1$  for  $K$  leaves  $m$  as a non-trivial free parameter in (4.10).

## 5. IDENTITIES OF THE ROGERS-RAMANUJAN-SLATER TYPE

We next prove a number of identities of the Rogers-Ramanujan-Slater type. We believe these to be new. We first prove two general transformations.

**Corollary 10.** For  $q$  and  $x$  inside the unit disc, and integers  $a > 0$  and  $b$ ,

$$(5.1) \quad \sum_{n=0}^{\infty} \frac{(q\sqrt{xyz}, -q\sqrt{xyz}, y, z; q)_n x^n q^{(an^2+bn)/2}}{(\sqrt{xyz}, -\sqrt{xyz}, qxy, qxz; q)_n (-q^{(a+b)/2}; q^a)_n} \\ = \frac{(1-xy)(1-xz)}{(1-x)(1-xyz)} \left( 1 - q^{(a-b)/2} \sum_{n=1}^{\infty} \frac{(y, z; q)_n x^n q^{(an^2+(b-2a)n)/2}}{(xy, xz; q)_n (-q^{(a+b)/2}; q^a)_n} \right).$$

*Proof.* In Corollary 1 set  $\alpha_0 = 1$  and, for  $n > 0$ ,

$$\alpha_n = \frac{q^{(an^2+bn)/2}}{(-q^{(a+b)/2}; q^a)_n} - \frac{q^{(a(n-1)^2+b(n-1))/2}}{(-q^{(a+b)/2}; q^a)_{n-1}} = -q^{(a-b)/2} \frac{q^{(an^2+(b-2a)n)/2}}{(-q^{(a+b)/2}; q^a)_n}.$$

□

**Corollary 11.** For  $q$  and  $x$  inside the unit disc, and integers  $a > 0$  and  $b$ ,

$$(5.2) \quad \sum_{n=0}^{\infty} \frac{(q\sqrt{xyz}, -q\sqrt{xyz}, y, z; q)_n x^n q^{an^2+bn}}{(\sqrt{xyz}, -\sqrt{xyz}, qxy, qxz; q)_n} = \frac{(1-xy)(1-xz)}{(1-x)(1-xyz)} \\ \times \left( 1 - q^{(a-b)} \sum_{n=1}^{\infty} \frac{(y, z; q)_n x^n q^{an^2+(b-2a)n} (1 - q^{2an+b-a})}{(xy, xz; q)_n} \right).$$

*Proof.* In Corollary 1 set  $\alpha_0 = 1$  and, for  $n > 0$ ,

$$\alpha_n = q^{an^2+bn} - q^{a(n-1)^2+b(n-1)} = -q^{an^2+(b-2a)n+a-b} (1 - q^{2an+b-a}).$$

□

**Corollary 12.**

$$(5.3) \quad \sum_{n=0}^{\infty} \frac{(1 + q^{-2n+3})q^{n^2+6n}}{(q^4; q^4)_n} = \frac{1}{(q^2, q^3; q^5)_{\infty} (-q^2; q^2)_{\infty}}.$$

$$(5.4) \quad \sum_{n=0}^{\infty} \frac{(1 + q^{-2n+1})q^{n^2+4n}}{(q^4; q^4)_n} = \frac{1}{(q, q^4; q^5)_{\infty} (-q^2; q^2)_{\infty}}.$$

*Proof.* In (5.2), set  $z = 0$ , replace  $x$  by  $x/y$  and let  $y \rightarrow \infty$  to get

$$(5.5) \quad \sum_{n=0}^{\infty} \frac{(-x)^n q^{an^2+bn+n(n-1)/2}}{(xq; q)_n} = (1-x) \\ \times \left( 1 - q^{(a-b)} \sum_{n=1}^{\infty} \frac{(-x)^n q^{an^2+(b-2a)n+n(n-1)/2} (1 - q^{2an+b-a})}{(x; q)_n} \right).$$

Next, let  $a = -1/4$ ,  $b = 1$ , replace  $q$  by  $q^4$  and let  $x \rightarrow 1$  to get

$$\sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2+2n}}{(q^4; q^4)_n} = -q^{-5} \sum_{n=1}^{\infty} \frac{(-1)^n q^{n^2+4n} (1 - q^{-2n+5})}{(q^4; q^4)_{n-1}}.$$

Replace  $q$  by  $-q$ , re-index the right side by replacing  $n$  by  $n + 1$  and (5.3) follows from the following identity of Rogers ([17], page 331):

$$\sum_{n=0}^{\infty} \frac{q^{n^2+2n}}{(q^4; q^4)_n} = \frac{1}{(q^2, q^3; q^5)_{\infty} (-q^2; q^2)_{\infty}}.$$

The identity at (5.4) follows similarly, using instead  $a = -1/4$ ,  $b = 1/2$  in (5.5) and employing another identity of Rogers ([17], page 330):

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q^4; q^4)_n} = \frac{1}{(q, q^4; q^5)_{\infty} (-q^2; q^2)_{\infty}}.$$

□

**Corollary 13.**

$$(5.6) \quad \sum_{n=0}^{\infty} \frac{(b, q^3/b; q)_n q^{n(n+1)/2}}{(q^2; q^2)_{n+1} (q; q)_n} = \frac{(q^4/b, bq; q^2)_{\infty}}{(q; q)_{\infty}}.$$

$$(5.7) \quad \sum_{n=0}^{\infty} \frac{(1 - q^{2n+1})(-q^3; q^2)_n q^{n^2}}{(q^2; q^2)_n} = \frac{1}{(q^3, q^4, q^5; q^8)_{\infty}}.$$

$$(5.8) \quad \sum_{n=0}^{\infty} \frac{(1 - q^{2n-1})(-q^3; q^2)_n q^{n^2-2n}}{(q^2; q^2)_n} = \frac{1}{(q, q^4, q^7; q^8)_{\infty}}.$$

$$(5.9) \quad 1 + \sum_{n=1}^{\infty} \frac{(-q; q)_n q^{(n^2-n)/2}}{(q; q)_{n-1}} = \frac{(-1; q)_{\infty} (-q^6, -q^{10}, q^{16}; q^{16})_{\infty}}{(q^4; q^4)_{\infty}}.$$

$$(5.10) \quad -1 + \sum_{n=1}^{\infty} \frac{(-q; q)_n q^{(n^2-n)/2}}{(q; q)_{n-1}} = q \frac{(-1; q)_{\infty} (-q^2, -q^{14}, q^{16}; q^{16})_{\infty}}{(q^4; q^4)_{\infty}}.$$

*Proof.* In (4.1), let  $z = 0$ , replace  $x$  by  $x/y$  and let  $y \rightarrow \infty$  to get

$$\sum_{n=0}^{\infty} \frac{(aq, bq; q)_n (-x)^n q^{n(n-1)/2}}{(qx, abq, q; q)_n} = (1-x) \sum_{n=0}^{\infty} \frac{(a, b; q)_n (-xq)^n q^{n(n-1)/2}}{(x, abq, q; q)_n}.$$

Then set  $x = -q$ ,  $a = b/q$  and then use Andrews'  $q$ -Bailey identity,

$$\sum_{n=0}^{\infty} \frac{(b, q/b; q)_n c^n q^{n(n-1)/2}}{(c; q)_n (q^2; q^2)_n} = \frac{(cq/b, bc; q^2)_{\infty}}{(c; q)_{\infty}}$$

with  $c = q^2$ , to sum the right side. Finally, replace  $b$  by  $b/q$  and (5.6) follows after a slight manipulation.

For the remaining identities, in (4.1) replace  $x$  by  $x/y$ , let  $y \rightarrow \infty$  and then set  $z = x$  and  $b = 0$  to get

$$\sum_{n=0}^{\infty} \frac{(1 + xq^n)(aq; q)_n (-x)^n q^{n(n-1)/2}}{(q; q)_n} = \sum_{n=0}^{\infty} \frac{(a; q)_n (-x)^n q^{n(n+1)/2}}{(q; q)_n}.$$

For (5.7) and (5.8), replace  $q$  by  $q^2$ , set  $a = -q$  and, respectively,  $x = -q$  and  $x = -1/q$ , and use the Göllnitz-Gordon-Slater identities ([12], [13], [22])

$$(5.11) \quad \sum_{n=0}^{\infty} \frac{q^{n^2+2n}(-q; q^2)_n}{(q^2; q^2)_n} = \frac{1}{(q^3; q^8)_{\infty}(q^4; q^8)_{\infty}(q^5; q^8)_{\infty}},$$

$$\sum_{n=0}^{\infty} \frac{q^{n^2}(-q; q^2)_n}{(q^2; q^2)_n} = \frac{1}{(q; q^8)_{\infty}(q^4; q^8)_{\infty}(q^7; q^8)_{\infty}},$$

to sum the respective right sides.

For (5.9), set  $a = x = -1$  and use the following identity of Gessel and Stanton ([11], page 196)

$$1 + \sum_{n=1}^{\infty} \frac{(-q; q)_{n-1} q^{(n^2+n)/2}}{(q; q)_n} = \frac{(-q; q)_{\infty}(-q^6, -q^{10}, q^{16}; q^{16})_{\infty}}{(q^4; q^4)_{\infty}}.$$

to sum the resulting right side. The identity at (5.10) follows similarly, again with  $a = x = -1$ , upon using another identity of Gessel and Stanton ([11], page 196)

$$\sum_{n=0}^{\infty} \frac{(-q; q)_n q^{(n^2+3n)/2}}{(q; q)_{n+1}} = \frac{(-q; q)_{\infty}(-q^2, -q^{14}, q^{16}; q^{16})_{\infty}}{(q^4; q^4)_{\infty}}.$$

□

#### Corollary 14.

$$(5.12) \quad \sum_{n=0}^{\infty} \frac{(-q^2; q^2)_{n+1} q^{n^2+2n}}{(q; q)_{2n+3}} = 2(-q^2, -q^{14}, q^{16}; q^{16})_{\infty} \frac{(-q; q^2)_{\infty}}{(q^2; q^2)_{\infty}} - \frac{1}{1-q}.$$

$$(5.13) \quad \sum_{n=0}^{\infty} \frac{(-q^2; q^2)_n q^{n^2}}{(q; q)_{2n+1}} = 2 \frac{(q^2, q^{14}, q^{16}; q^{16})_{\infty} (q^{12}, q^{20}; q^{32})_{\infty}}{(q; q)_{\infty}} - 1.$$

*Proof.* We use (5.1) to prove these identities. First, let  $z \rightarrow 0$  and replace  $q$  with  $q^2$  to get

$$(5.14) \quad \sum_{n=0}^{\infty} \frac{(y; q^2)_n x^n q^{an^2+bn}}{(q^2 xy; q^2)_n (-q^{a+b}; q^{2a})_n}$$

$$= \frac{(1-xy)}{(1-x)} \left( 1 - q^{a-b} \sum_{n=1}^{\infty} \frac{(y; q^2)_n x^n q^{an^2+(b-2a)n}}{(xy; q^2)_n (-q^{a+b}; q^{2a})_n} \right).$$

For (5.12), set  $a = 1$ ,  $b = 2$ ,  $y = -q^2$ , and  $x = -1$ . Replace  $q$  with  $-q$ , divide both sides by  $(1-q)(1-q^2)$  and use Slater's identity **69** to sum the resulting left side:

$$\sum_{n=0}^{\infty} \frac{(-q^2; q^2)_n q^{n^2+2n}}{(q; q)_{2n+2}} = (-q^2, -q^{14}, q^{16}; q^{16})_{\infty} \frac{(-q; q^2)_{\infty}}{(q^2; q^2)_{\infty}}.$$

The result follows after some slight manipulation.

The proof of (5.13) is similar, except we set  $a = 1$ ,  $b = 0$ ,  $y = -1$ , and  $x = -1$ , replace  $q$  with  $-q$ , and use Slater's identity **121**:

$$1 + \sum_{n=1}^{\infty} \frac{(-q^2; q^2)_{n-1} q^{n^2}}{(q; q)_{2n}} = \frac{(q^2, q^{14}, q^{16}; q^{16})_{\infty} (q^{12}, q^{20}; q^{32})_{\infty}}{(q; q)_{\infty}}.$$

□

## 6. REPRESENTATION OF FALSE THETA SERIES AS BASIC HYPERGEOMETRIC SERIES

In this section we derive some new representations of the false theta series  $\sum_{n=0}^{\infty} (-1)^n q^{n(n+1)/2}$  and  $\sum_{n=0}^{\infty} q^{n(3n+1)/2} (1 - q^{2n+1})$ , as basic hypergeometric series.

On page 13 of the Lost Notebook [16] (see also [3, page 229]), Ramanujan recorded the following identity (amongst others in a similar vein):

$$(6.1) \quad \sum_{n=0}^{\infty} \frac{(q; q^2)_n (-1)^n q^{n^2+n}}{(-q; q)_{2n+1}} = \sum_{n=0}^{\infty} (-1)^n q^{n(n+1)/2}.$$

On page 37 of the Lost Notebook, he recorded the identities

$$(6.2) \quad \begin{aligned} \sum_{n=0}^{\infty} q^{n(3n+1)/2} (1 - q^{2n+1}) &= \sum_{n=0}^{\infty} \frac{q^{2n^2+n}}{(-q; q)_{2n+1}} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n+1)/2}}{(-q; q)_n}. \end{aligned}$$

The identity that follows from equating the left side to the second right side above also follows as a special case of a more general identity first stated by Rogers [18].

We use these identities in conjunction with (5.14) to prove the following.

**Corollary 15.**

$$(6.3) \quad 1 - \sum_{n=0}^{\infty} \frac{(q; q^2)_n (-1)^n q^{n^2-n}}{(-1; q)_{2n+1}} = \sum_{n=0}^{\infty} (-1)^n q^{n(n+1)/2}.$$

$$(6.4) \quad \frac{2}{1+q} - \sum_{n=0}^{\infty} \frac{q^{2n^2+3n}}{(-q; q)_{2n+1} (1 + q^{2n+3})} = \sum_{n=0}^{\infty} q^{n(3n+1)/2} (1 - q^{2n+1}).$$

$$(6.5) \quad \frac{1}{2} + \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n+1)/2}}{(-1; q)_{n+2}} = \sum_{n=0}^{\infty} q^{n(3n+1)/2} (1 - q^{2n+1}).$$

*Proof.* For (6.3), set  $a = b = 1$ ,  $y = q$  and  $x = -1$  in (5.14). Then divide both sides of the resulting identity by  $1 + q$ , so that the left side becomes the left side of (6.1). The result follows after re-indexing the resulting sum on the right side, together with a little manipulation.

For (6.4), replace  $x$  with  $x/y$  in (5.14) and let  $y \rightarrow \infty$  to get

$$(6.6) \quad \sum_{n=0}^{\infty} \frac{(-x)^n q^{(a+1)n^2+(b-1)n}}{(q^2x; q^2)_n (-q^{a+b}; q^{2a})_n} \\ = (1-x) \left( 1 - q^{a-b} \sum_{n=1}^{\infty} \frac{(-x)^n q^{(a+1)n^2+(b-2a-1)n}}{(x; q^2)_n (-q^{a+b}; q^{2a})_n} \right).$$

Then set  $a = 1$ ,  $b = 2$ ,  $x = -1$ , and divide both sides by  $1 + q$  so that the left side becomes the first right side of (6.2). The result again follows, upon re-indexing the sum on the right side.

To get (6.3), set  $y = 0$  in (5.14), then  $a = b = 1/2$  and  $x = -1$ , so the left side becomes the second right side in (6.2). The result likewise follows after re-indexing the resulting sum on the right side.  $\square$

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