# Some Applications of a Bailey-type Transformation

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# SOME APPLICATIONS OF A BAILEY-TYPE TRANSFORMATION

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Abstract. If $k$ is set equal to $aq$ in the definition of a WP Bailey pair,

$$\beta_n(a,k)=\sum_{j=0}^{n}\frac{(k/a)_{n-j}(k)_{n+j}}{(q)_{n-j}(aq)_{n+j}}\alpha_j(a,k),$$

this equation reduces to $\beta_n=\sum_{j=0}^{n}\alpha_j$.

This seemingly trivial relation connecting the $\alpha_n$'s with the $\beta_n$'s has some interesting consequences, including several basic hypergeometric summation formulae, a connection to the Prouhet-Tarry-Escott problem, some new identities of the Rogers-Ramanujan-Slater type, some new expressions for false theta series as basic hypergeometric series, and new transformation formulae for poly-basic hypergeometric series.

## 1. Introduction

We begin by recalling a construction of Andrews [1]. If a pair of sequences $(\alpha_n(a,k),\ \beta_n(a,k))$ satisfy

$$\beta_n(a,k)=\sum_{j=0}^{n}\frac{(k/a)_{n-j}(k)_{n+j}}{(q)_{n-j}(aq)_{n+j}}\alpha_j(a,k), \tag{1.1}$$

then so does the pair $(\alpha'_n(a,k),\ \beta'_n(a,k))$, where

$$\begin{aligned}\alpha'_n(a,k)&=\frac{(\rho_1,\rho_2)_n}{(aq/\rho_1,aq/\rho_2)_n}\left(\frac{k}{c}\right)^n\alpha_n(a,c),\\ \beta'_n(a,k)&=\frac{(k\rho_1/a,k\rho_2/a)_n}{(aq/\rho_1,aq/\rho_2)_n}\\ &\quad\times\sum_{j=0}^{n}\frac{(1-cq^{2j})(\rho_1,\rho_2)_j(k/c)_{n-j}(k)_{n+j}}{(1-c)(k\rho_1/a,k\rho_2/a)_n(q)_{n-j}(qc)_{n+j}}\left(\frac{k}{c}\right)^j\beta_j(a,c),\end{aligned} \tag{1.2}$$

with $c=k\rho_1\rho_2/aq$. A pair of sequences satisfying (1.1) is termed a *WP-Bailey pair*. If $k=0$, the pair of sequences become what is termed a *Bailey pair relative to a*.

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Bailey [4, 5] used the $q$-Gauss sum,

$$
{}_2\phi_1(a, b; c; q, c/ab) = \frac{(c/a, c/b; q)_\infty}{(c, c/ab; q)_\infty}, \tag{1.3}
$$

to get that, if $(\alpha_n, \beta_n)$ are a Bailey pair relative to $a$, then

$$
\sum_{n=0}^{\infty} (y, z; q)_n \left(\frac{aq}{yz}\right)^n \beta_n = \frac{(aq/y, aq/z; q)_\infty}{(aq, aq/yz; q)_\infty} \sum_{n=0}^{\infty} \frac{(y, z; q)_n}{(aq/y, aq/z; q)_n} \left(\frac{x}{yz}\right)^n \alpha_n. \tag{1.4}
$$

Slater, in [21] and [22], subsequently used this transformation of Bailey to derive 130 identities of the Rogers-Ramanujan type.

The first major variations in Bailey's construct at (1.4) appear to be due to Bressoud [7]. Another variation was given by Singh in [20]. All of these variations were put in a more formal setting by Andrews in [1], where he introduced the generalization of the standard Bailey pair defined above.

In the same paper Andrews also described a second way to construct a new WP-Bailey pair from a given pair. These two constructions allowed a "tree" of WP-Bailey pairs to be generated from a single WP-Bailey pair. These two branches of the WP-Bailey tree were further investigated by Andrews and Berkovich in [2]. Spiridonov [23] derived an elliptic generalization of Andrews first WP-Bailey chain, and Warnaar [25] added four new branches to the WP-Bailey tree, two of which had generalizations to the elliptic level. More recently, and motivated in part by the papers above, Liu and Ma [14] introduced the idea of a general WP-Bailey chain (as a solution to a system of linear equations), and added one new branch to the WP-Bailey tree.

As we might expect, Andrews generalization of a Bailey pair leads to a generalization of (1.4). Indeed Andrews WP-Bailey chain at (1.2) can easily be shown to imply the following result (substitute the expression for $\alpha'_n(a, k)$ in (1.1), set the two expressions for $\beta'_n(a, k)$ equal, and let $n \to \infty$). Note that setting $k = 0$ recovers Bailey's transformation at (1.4). (We initially derived (1.6) in a way similar to Bailey's derivation of (1.4), before realizing that it followed from Andrews' construction (1.2).)

**Theorem 1.** *Under suitable convergence conditions, if* $(\alpha_n(a, k), \beta_n(a, k))$ *satisfy*

$$
\beta_n(a, k) = \sum_{j=0}^{n} \frac{(k/a)_{n-j}(k)_{n+j}}{(q)_{n-j}(aq)_{n+j}} \alpha_j(a, k), \tag{1.5}
$$

*then*

$$
\sum_{n=0}^{\infty} \frac{(1 - kq^{2n})(\rho_1, \rho_2; q)_n}{(1 - k)(kq/\rho_1, kq/\rho_2; q)_n} \left(\frac{aq}{\rho_1\rho_2}\right)^n \beta_n(a, k) = \tag{1.6}
$$

$$
\frac{(kq, kq/\rho_1\rho_2, aq/\rho_1, aq/\rho_2; q)_\infty}{(kq/\rho_1, kq/\rho_2, aq/\rho_1\rho_2, aq; q)_\infty} \sum_{n=0}^{\infty} \frac{(\rho_1, \rho_2; q)_n}{(aq/\rho_1, aq/\rho_2; q)_n} \left(\frac{aq}{\rho_1\rho_2}\right)^n \alpha_n(a, k).
$$

In the present paper we investigate what at first glance may appear to be a trivial special case of Theorem 1.

**Corollary 1.** *If* $\beta_n = \sum_{r=0}^{n} \alpha_r$*, then assuming both series converge,*

(1.7)
$$\sum_{n=0}^{\infty} \frac{(q\sqrt{xyz}, -q\sqrt{xyz}, y, z; q)_n x^n \beta_n}{(\sqrt{xyz}, -\sqrt{xyz}, qxy, qxz; q)_n} = \frac{(1-xy)\,(1-xz)}{(1-x)\,(1-xyz)} \sum_{n=0}^{\infty} \frac{(y, z; q)_n x^n \alpha_n}{(xy, xz; q)_n}.$$

*Proof.* Let $k = xyz$, $a = xyz/q$, $\rho_1 = y$ and $\rho_2 = z$ in Theorem 1. □

This seemingly trivial relation connecting the $\alpha_n$'s with the $\beta_n$'s has some interesting consequences, including several basic hypergeometric summation formulae, a connection to the Prouhet-Tarry-Escott problem, some new identities of the Rogers-Ramanujan-Slater type, some new expressions for false theta series as basic hypergeometric series, and new transformation formulae for poly-basic hypergeometric series.

We employ the usual notations. Let $a$ and $q$ be complex numbers, with $|q| < 1$ unless otherwise stated. Then

$$\begin{aligned}
&(a)_0 = (a;q)_0 := 1, \qquad (a)_n = (a;q)_n := \prod_{j=0}^{n-1}(1-aq^j), \text{ for } n \in \mathbb{N},\\
&(a_1;q)_n(a_2;q)_n \dots (a_k;q)_n = (a_1, a_2, \dots, a_k;q)_n,\\
&(a;q)_\infty := \prod_{j=0}^{\infty}(1-aq^j),\\
&(a_1;q)_\infty(a_2;q)_\infty \dots (a_k;q)_\infty = (a_1, a_2, \dots, a_k;q)_\infty.
\end{aligned}$$

An ${}_r\phi_s$ basic hypergeometric series is defined by

$${}_r\phi_s \left[ \begin{matrix} a_1, a_2, \dots, a_r \\ b_1, \dots, b_s \end{matrix} ; q, x \right] = \sum_{n=0}^{\infty} \frac{(a_1;q)_n(a_2;q)_n \dots (a_r;q)_n}{(q;q)_n(b_1;q)_n \dots (b_s;q)_n} \left((-1)^n q^{n(n-1)/2}\right)^{s+1-r} x^n.$$

For future use we also recall the $q$-binomial theorem,

$$\sum_{n=0}^{\infty} \frac{(a;q)_n}{(q;q)_n} z^n = \frac{(az;q)_\infty}{(z;q)_\infty}. \tag{1.8}$$

## 2. Various Summation Formulae for Basic Hypergeometric series

We next derive a number of transformation formulae for basic hypergeometric series, transformations that give rise to summation formulae for particular choices of the parameters.

**Corollary 2.** *For $q$ and $x$ inside the unit disc,*

$$\sum_{n=0}^{\infty} \frac{(q\sqrt{xyz}, -q\sqrt{xyz}, y, z; q)_{2n} x^{2n}}{(\sqrt{xyz}, -\sqrt{xyz}, qxy, qxz; q)_{2n}} = \frac{(1-xy)\,(1-xz)}{(1-x)\,(1-xyz)} \sum_{n=0}^{\infty} \frac{(y, z; q)_n (-x)^n}{(xy, xz; q)_n}. \tag{2.1}$$

$$\sum_{n=0}^{\infty} \frac{(1-q^{2n+1}/x)(q/x^2;q)_{2n} x^{2n}}{(q;q)_{2n+1}} = \frac{1}{1+x} \frac{(q/x;q)_\infty}{(x;q)_\infty}, \quad x \neq 0. \tag{2.2}$$

*Proof.* In Corollary 1 let $\alpha_r = (-1)^r$ to get (2.1). Then set $y = q/x$, $z = q/x^2$, apply (1.8) to the right side, replace $x$ by $-x$ and (2.2) follows. □

**Corollary 3.** *For $q$ and $x$ inside the unit disc,*

$$\sum_{n=0}^{\infty} \frac{(q\sqrt{xyz}, -q\sqrt{xyz}, y, z; q)_n x^n (n+1)}{(\sqrt{xyz}, -\sqrt{xyz}, qxy, qxz; q)_n} = \frac{(1-xy)\,(1-xz)}{(1-x)\,(1-xyz)} \sum_{n=0}^{\infty} \frac{(y, z; q)_n x^n}{(xy, xz; q)_n}. \tag{2.3}$$

$$\sum_{n=0}^{\infty} \frac{(1+q^{n+1}/x)(q/x^2;q)_n x^n (n+1)}{(q;q)_{n+1}} = \frac{1}{1-x} \frac{(q/x;q)_\infty}{(x;q)_\infty}. \tag{2.4}$$

*Proof.* Set $\alpha_n = 1$ in Corollary 1 to get (2.3). The identity at (2.4) follows from (2.3) upon setting $y = q/x^2$, $z = q/x$, using (1.8) to sum the right side and then simplifying. □

**Corollary 4.** *For $q$, $x$ and $u$ all inside the unit disc,*

$$\sum_{n=0}^{\infty} \frac{(1+q^{n+1}/x)(q/x^2;q)_n x^n (1-u^{n+1})}{(q;q)_{n+1}} = \frac{1-u}{1-x} \frac{(qu/x;q)_\infty}{(xu;q)_\infty}. \tag{2.5}$$

*Proof.* Set $\alpha_n = u^n$, $y = q/x$ and $z = q/x^2$. Now apply the $q$-binomial theorem (1.8) to the right side. □

**Corollary 5.**

$$ {}_5\phi_4 \left[ \begin{matrix} q\sqrt{xyz}, -q\sqrt{xyz}, y, z, cq \\ \sqrt{xyz}, -\sqrt{xyz}, qxy, qxz \end{matrix} ; q, x \right] = \frac{(1-xy)(1-xz)}{(1-x)(1-xyz)} \, {}_3\phi_2 \left[ \begin{matrix} y, z, c \\ xy, xz \end{matrix} ; q, xq \right]. \tag{2.6}$$

$$ {}_3\phi_2 \left[ \begin{matrix} -qxy, y, x \\ -xy, qx^2y \end{matrix} ; q, x \right] = \frac{1}{1+xy} \frac{(x^2, qxy; q)_\infty}{(qx^2y, x; q)_\infty}. \tag{2.7}$$

*Proof.* We define $\alpha_0 = 1$, and for $n > 0$,

$$\alpha_n = \frac{(cq;q)_n}{(q;q)_n} - \frac{(cq;q)_{n-1}}{(q;q)_{n-1}} = \frac{(c;q)_n}{(q;q)_n}q^n.$$

Substitution into (1.7) immediately gives (2.6). Equation (2.7) follows upon letting $c = x/q$, $z = xy$ and using (1.3) to sum the resulting right side and simplifying. □

## 3. Transformation Formulae for basic- and polybasic Hypergeometric Series

In contrast to the situation with basic hypergeometric series, most (possibly all) summation formulae for poly-basic hypergeometric series arise because the series involved telescope. This means that the terms in such an identity may be inserted in (1.7) to produce a transformation formula for polybasic hypergeometric series containing an additional base. Setting all the bases equal to $q^m$, for some integer $m$, then gives a transformation formula for basic hypergeometric series. We give one example in the next corollary, which contains a transformation formula connecting polybasic hypergeometric series with five independent bases.

**Corollary 6.** *Let $P$, $p$, $Q$, $q$, $R$ and $x$ all lie inside the unit disc, and let $a$, $b$, $c$, $y$ and $z$ be complex numbers such that the denominators below are bounded away from zero. Then*

$$\begin{aligned}(3.1)\quad &\sum_{n=0}^{\infty} \frac{(q\sqrt{xyz}, -q\sqrt{xyz}, y, z; q)_n}{(\sqrt{xyz}, -\sqrt{xyz}, qxy, qxz; q)_n} \frac{(ap^2;p^2)_n \left(bP^2;P^2\right)_n}{\left(\frac{PQR}{p}; \frac{PQR}{p}\right)_n \left(\frac{apPQ}{cR}; \frac{pPQ}{R}\right)_n} \\ &\times \frac{(cR^2;R^2)_n \left(\frac{aQ^2}{bc};Q^2\right)_n}{\left(\frac{apQR}{bP}; \frac{pQR}{P}\right)_n \left(\frac{bcpPR}{Q}; \frac{pPR}{Q}\right)_n} x^n \\ &= \frac{(1-xy)(1-xz)}{(1-x)(1-xyz)} \times \\ &\sum_{n=0}^{\infty} \frac{(y,z;q)_n}{(xy,xz;q)_n} \frac{(1-ap^nP^nQ^nR^n)\left(1-b\frac{p^nP^n}{Q^nR^n}\right)\left(1-\frac{P^nQ^n}{cp^nR^n}\right)\left(1-\frac{ap^nQ^n}{bcP^nR^n}\right)}{(1-a)(1-b)\left(1-\frac{1}{c}\right)\left(1-\frac{a}{bc}\right)} \\ &\times \frac{(a;p^2)_n \left(b;P^2\right)_n}{\left(\frac{PQR}{p}; \frac{PQR}{p}\right)_n \left(\frac{apPQ}{cR}; \frac{pPQ}{R}\right)_n} \frac{(c;R^2)_n \left(\frac{a}{bc};Q^2\right)_n}{\left(\frac{apQR}{bP}; \frac{pQR}{P}\right)_n \left(\frac{bcpPR}{Q}; \frac{pPR}{Q}\right)_n} (xR^2)^n;\end{aligned}$$

$$\begin{aligned}(3.2)\quad &\sum_{n=0}^{\infty} \frac{(q\sqrt{xyz}, -q\sqrt{xyz}, y, z; q)_n}{(\sqrt{xyz}, -\sqrt{xyz}, qxy, qxz; q)_n} \frac{(aq^m, bq^m, cq^m, \frac{aq^m}{bc}; q^m)_n}{\left(\frac{a}{c}q^m, \frac{a}{b}q^m, bcq^m, q^m; q^m\right)_n} x^n \\ &= \frac{(1-xy)(1-xz)}{(1-x)(1-xyz)} \times\end{aligned}$$

$$\sum_{n=0}^{\infty} \frac{(y,z;q)_n}{(xy,xz;q)_n} \frac{(q^m\sqrt{a},-q^m\sqrt{a},a,b,c,\frac{a}{bc};q^m)_n}{\left(\sqrt{a},-\sqrt{a},\frac{a}{c}q^m,\frac{a}{b}q^m,bcq^m,q^m;q^m\right)_n}(xq^m)^n.$$

*Proof.* We use the special case $m=0$, $d=1$ of the identity of Subbarao and Verma labeled (2.2) in [24], namely,

$$\sum_{k=0}^{n} \frac{(1-ap^kP^kQ^kR^k)\left(1-b\frac{p^kP^k}{Q^kR^k}\right)\left(1-\frac{P^kQ^k}{cp^kR^k}\right)\left(1-\frac{ap^kQ^k}{bcP^kR^k}\right)}{(1-a)(1-b)\left(1-\frac{1}{c}\right)\left(1-\frac{a}{bc}\right)} \tag{3.3}$$
$$\times \frac{(a;p^2)_k\left(b;P^2\right)_k}{\left(\frac{PQR}{p};\frac{PQR}{p}\right)_k\left(\frac{apPQ}{cR};\frac{pPQ}{R}\right)_k} \frac{(c;R^2)_k\left(\frac{a}{bc};Q^2\right)_k}{\left(\frac{apQR}{bP};\frac{pQR}{P}\right)_k\left(\frac{bcpPR}{Q};\frac{pPR}{Q}\right)_k} R^{2k}$$
$$= \frac{(ap^2;p^2)_n\left(bP^2;P^2\right)_n(cR^2;R^2)_n\left(\frac{aQ^2}{bc};Q^2\right)_n}{\left(\frac{PQR}{p};\frac{PQR}{p}\right)_n\left(\frac{apPQ}{cR};\frac{pPQ}{R}\right)_n\left(\frac{apQR}{bP};\frac{pQR}{P}\right)_n\left(\frac{bcpPR}{Q};\frac{pPR}{Q}\right)_n},$$

and then in (1.7) let $\alpha_i$ be the $i$-th term in the sum above, and let $\beta_n$ be the quantity on the right side above.

The identity at (3.2) follows upon setting $P=Q=p=R=q^{m/2}$ and simplifying. $\square$

## 4. A Connection with the Prouhet-Tarry-Escott Problem

We begin with a simple example.

**Corollary 7.**

$$ {}_6\phi_5\left[\begin{matrix} q\sqrt{xyz},-q\sqrt{xyz},y,z,aq,bq \\ \sqrt{xyz},-\sqrt{xyz},qxy,qxz,abq\end{matrix};q,x\right] \tag{4.1}$$
$$= \frac{(1-xy)(1-xz)}{(1-x)(1-xyz)}\,{}_4\phi_3\left[\begin{matrix} y,z,a,b \\ xy,xz,abq\end{matrix};q,xq\right].$$

*Proof.* This time, in Corollary 1, define $\alpha_0=1$, and for $n>0$,

$$\alpha_n = \frac{(aq,bq;q)_n}{(abq,q;q)_n} - \frac{(aq,bq;q)_{n-1}}{(abq,q;q)_{n-1}} = \frac{(a,b;q)_nq^n}{(abq,q;q)_n}.$$

The result follows as above. $\square$

The telescoping approach used in Corollary 7 can be generalized in one direction. We have the following result.

**Proposition 1.** *Let $x$, $y$ and $q$ be complex numbers with $|x|$, $|q|<1$. Suppose $a_1,a_2,\dots,a_m$ are non-zero complex numbers and let $b_1,b_2,\dots,b_{m-1}$ satisfy*

$$(z-1)\prod_{i=1}^{m-1}(z-b_i) = \prod_{i=1}^{m}(z-a_i) - \prod_{i=1}^{m}(1-a_i). \tag{4.2}$$

*Suppose further that* $b_i \neq 0$, *for* $1 \leq i \leq m-1$. *Then*

$$(4.3) \quad {}_{m+4}\phi_{m+3}\left[\begin{matrix} q\sqrt{xyz},\, -q\sqrt{xyz},\, y,\, z,\, a_1q,\, \dots,\, a_{m-1}q,\, a_mq \\ \sqrt{xyz},\, -\sqrt{xyz},\, qxy,\, qxz,\, b_1q,\, \dots,\, b_{m-1}q \end{matrix}; q,\, x\right]$$
$$= \frac{(1-xy)(1-xz)}{(1-x)(1-xyz)}\, {}_{m+2}\phi_{m+1}\left[\begin{matrix} y,\, z,\, a_1,\, \dots,\, a_{m-1},\, a_m \\ xy,\, xz,\, b_1q,\, \dots,\, b_{m-1}q \end{matrix}; q,\, xq^m\right].$$

*Proof.* Define $\alpha_0 = 1$, and for $n \geq 1$, set

$$\alpha_n = \frac{(a_1q, a_2q, \dots, a_{m-1}q, a_mq; q)_n}{(b_1q, b_2q, \dots, b_{m-1}q,\, q; q)_n} - \frac{(a_1q, a_2q, \dots, a_{m-1}q, a_mq; q)_{n-1}}{(b_1q, b_2q, \dots, b_{m-1}q,\, q; q)_{n-1}}.$$

By (4.2),

$$\alpha_n = \frac{(a_1, a_2, \dots, a_{m-1}, a_m; q)_n}{(b_1q, b_2q, \dots, b_{m-1}q,\, q; q)_n}\, q^{mn}$$

and clearly

$$(4.4) \qquad \beta_n = \sum_{r=0}^{n} \alpha_r = \frac{(a_1q, a_2q, \dots, a_{m-1}q, a_mq; q)_n}{(b_1q, b_2q, \dots, b_{m-1}q,\, q; q)_n}.$$

The result follows from Corollary 1. □

The fundamental theorem of algebra guarantees that there is no shortage of sets of complex numbers $a_1, a_2, \dots, a_m$, $b_1, b_2, \dots, b_{m-1}$ satisfying (4.2), but for $m > 4$ it is still a problem to find explicit examples. However, a related problem in number theory provides solutions for $m \leq 10$ and $m = 12$.

The *Prouhet-Tarry-Escott* problem asks for two distinct multisets of integers $A = \{a_1, ..., a_m\}$ and $B = \{b_1, ..., b_m\}$ such that

$$(4.5) \qquad \sum_{i=1}^{m} a_i^e = \sum_{i=1}^{m} b_i^e, \text{ for } e = 1, 2, \dots, k,$$

for some integer $k < m$. If $k = m-1$, such a solution is called *ideal.* We write

$$(4.6) \qquad \{a_1, ..., a_m\} \stackrel{k}{=} \{b_1, ..., b_m\}$$

to denote a solution to the Prouhet-Tarry-Escott problem.

The connection between the Prouhet-Tarry-Escott problem and the problem mentioned above is contained in the following proposition (see [6], page 2065).

**Proposition 2.** *The multisets* $A = \{a_1, ..., a_m\}$ *and* $B = \{b_1, ..., b_m\}$ *form an ideal solution to the Prouhet-Tarry-Escott problem if and only if*

$$\prod_{i=1}^{m}(z - a_i) - \prod_{i=1}^{m}(z - b_i) = C,$$

*for some constant* $C$.

Note that the fact that $b_m = 1$ is not a problem, since if

$$\{a_1, ..., a_m\} \overset{m-1}{=} \{b_1, ..., b_m\},$$

then

$$\{Ma_1 + K, ..., Ma_m + K\} \overset{m-1}{=} \{Mb_1 + K, ..., Mb_m + K\},$$

for constants $M$ and $K$ (see Lemma 1 in [8], for example).

Parametric ideal solutions are known for $m = 1, \dots, 8$ and particular numerical solutions are known for $m = 9, 10$ and 12. Although every ideal solution to the Prouhet-Tarry-Escott problem gives rise to a transformation between basic hypergeometric series, we will consider just one example. Note also that it is not necessary, for our purposes, that the $a_i$'s and $b_i$'s be integers. As above, we assume $x$, $y$ and $q$ are complex numbers, with $|x|, |q| < 1$.

**Corollary 8.** *Let $m$ and $n$ be non-zero complex numbers. Set*

$$\begin{aligned}
&a_1 = -3m^2 + 7nm - 2n^2 + 1, && b_1 = -3m^2 + 8nm + n^2 + 1,\\
&a_2 = -2m^2 + 8nm + 2n^2 + 1, && b_2 = -2m^2 + 3nm - 3n^2 + 1,\\
&a_3 = -m^2 - n^2 + 1, && b_3 = -m^2 + 10nm - n^2 + 1,\\
&a_4 = 2m^2 + 3nm + n^2 + 1, && b_4 = 2m^2 + 2nm - 2n^2 + 1,\\
&a_5 = m^2 + 2nm - 3n^2 + 1, && b_5 = m^2 + 7nm + 2n^2 + 1,\\
&a_6 = 10mn + 1. &&
\end{aligned} \tag{4.7}$$

*Then*

$$\begin{aligned}
&{}_{10}\phi_9\left[\begin{matrix} q\sqrt{xyz},\ -q\sqrt{xyz},\ y,\ z,\ a_1q,\ a_2q,\ a_3q,\ a_4q,\ a_5q,\ a_6q \\ \sqrt{xyz},\ -\sqrt{xyz},\ qxy,\ qxz,\ b_1q,\ b_2q,\ b_3q,\ b_4q,\ b_5q \end{matrix}; q,\ x\right]\\
&\qquad = \frac{(1-xy)(1-xz)}{(1-x)(1-xyz)}\,{}_8\phi_7\left[\begin{matrix} y,\ z,\ a_1,\ a_2,\ a_3,\ a_4,\ a_5,\ a_6 \\ xy,\ xz,\ b_1q,\ b_2q,\ b_3q,\ b_4q,\ b_5q \end{matrix}; q,\ xq^6\right].
\end{aligned} \tag{4.8}$$

*Proof.* We have from page 629–30 and Lemma 1 in [8], that if

$$\begin{aligned}
&a_1 = -5m^2 + 4nm - 3n^2 + K, && b_1 = -5m^2 + 6nm + 3n^2 + K,\\
&a_2 = -3m^2 + 6nm + 5n^2 + K, && b_2 = -3m^2 - 4nm - 5n^2 + K,\\
&a_3 = -m^2 - 10nm - n^2 + K, && b_3 = -m^2 + 10nm - n^2 + K,\\
&a_4 = 5m^2 - 4nm + 3n^2 + K, && b_4 = 5m^2 - 6nm - 3n^2 + K,\\
&a_5 = 3m^2 - 6nm - 5n^2 + K, && b_5 = 3m^2 + 4nm + 5n^2 + K,\\
&a_6 = m^2 + 10nm + n^2 + K, && b_6 = m^2 - 10nm + n^2 + K,
\end{aligned} \tag{4.9}$$

then

$$\{a_1, a_2, a_3, a_4, a_5, a_6\} \overset{5}{=} \{b_1, b_2, b_3, b_4, b_5, b_6\}.$$

We set $b_6 = 1$, solve for $K$ and back-substitute in (4.9). We then replace $m$ by $m/\sqrt{2}$ and $n$ by $n/\sqrt{2}$. This leads to the values for the $a_i$'s and $b_i$'s given at (4.7) and the result follows, as before, from Proposition 1. □

We also note each ideal solution to the Prouhet-Tarry-Escott problem leads to an infinite summation formula, upon letting $n \to \infty$ in (4.4). We give one example.

**Corollary 9.** *Let $m$ be a non-zero complex number. Set*

$$\begin{aligned}\{a_i\}_{i=1}^{12} = \{&1+170m, 1+126m, 1+209m, 1+87m, 1+234m, 1+62m,\\ &1+275m, 1+21m, 1+288m, 1+8m, 1+299m, 1-3m\},\end{aligned}$$

$$\begin{aligned}\{b_i\}_{i=1}^{11} = \{&1+183m, 1+113m, 1+195m, 1+101m, 1+242m, 1+54m,\\ &1+269m, 1+27m, 1+294m, 1+2m, 1+296m\}.\end{aligned}$$

*Then*

$$\begin{aligned}&{}_{12}\phi_{11}\left[\begin{matrix}a_1,\, a_2,\, a_3,\, a_4,\, a_5,\, a_6,\, a_7,\, a_8, a_9, a_{10}, a_{11}, a_{12}\\ b_1q,\, b_2q,\, b_3q,\, b_4q,\, b_5q,\, b_6q, b_7q, b_9q, b_{10}q, b_{11}q\end{matrix}; q,\, q^{12}\right].\\ &= \frac{(a_1q, a_2q, a_3q, a_4q, a_5q, a_6q, a_7q, a_8q, a_9q, a_{10}q, a_{11}q, a_{12}q; q)_\infty}{(b_1q,\, b_2q,\, b_3q,\, b_4q,\, b_5q,\, b_6q, b_7q, b_9q, b_{10}q, b_{11}q, q; q)_\infty}\end{aligned} \tag{4.10}$$

*Proof.* We use a result of Nuutti Kuosa, Jean-Charles Meyrignac and Chen Shuwen (see [19]), namely, that if

(4.11)

$$\begin{aligned}A = \{&K+22m, K-22m, K+61m, K-61m, K+86m, K-86m,\\ &K+127m, K-127m, K+140m, K-140m, K+151m, K-151m\},\\ B = \{&K+35m, K-35m, K+47m, K-47m, K+94m, K-94m,\\ &K+121m, K-121m, K+146m, K-146m, K+148m, K-148m\},\end{aligned}$$

then

$$A \stackrel{11}{=} B.$$

□

Remark: Note that while the $K$ and $m$ are irrelevant in (4.11) in so far as finding integer solutions to the Prouhet-Tarry-Escott problem (since the solution derived another solution by scaling by $m$ and translating by $K$ is trivially equivalent to the original solution), solving $B_{12} = 1$ for $K$ leaves $m$ as a non-trivial free parameter in (4.10).

## 5. Identities of the Rogers-Ramanujan-Slater Type

We next prove a number of identities of the Rogers-Ramanujan-Slater type. We believe these to be new. We first prove two general transformations.

**Corollary 10.** *For $q$ and $x$ inside the unit disc, and integers $a > 0$ and $b$,*

$$\sum_{n=0}^{\infty} \frac{(q\sqrt{xyz}, -q\sqrt{xyz}, y, z; q)_n x^n q^{(an^2+bn)/2}}{(\sqrt{xyz}, -\sqrt{xyz}, qxy, qxz; q)_n (-q^{(a+b)/2}; q^a)_n} \tag{5.1}$$
$$= \frac{(1-xy)(1-xz)}{(1-x)(1-xyz)} \left(1 - q^{(a-b)/2} \sum_{n=1}^{\infty} \frac{(y, z; q)_n x^n q^{(an^2+(b-2a)n)/2}}{(xy, xz; q)_n (-q^{(a+b)/2}; q^a)_n}\right).$$

*Proof.* In Corollary 1 set $\alpha_0 = 1$ and, for $n > 0$,

$$\alpha_n = \frac{q^{(an^2+bn)/2}}{(-q^{(a+b)/2}; q^a)_n} - \frac{q^{(a(n-1)^2+b(n-1))/2}}{(-q^{(a+b)/2}; q^a)_{n-1}} = -q^{(a-b)/2} \frac{q^{(an^2+(b-2a)n)/2}}{(-q^{(a+b)/2}; q^a)_n}.$$

□

**Corollary 11.** *For $q$ and $x$ inside the unit disc, and integers $a > 0$ and $b$,*

$$\sum_{n=0}^{\infty} \frac{(q\sqrt{xyz}, -q\sqrt{xyz}, y, z; q)_n x^n q^{an^2+bn}}{(\sqrt{xyz}, -\sqrt{xyz}, qxy, qxz; q)_n} = \frac{(1-xy)(1-xz)}{(1-x)(1-xyz)} \tag{5.2}$$
$$\times \left(1 - q^{(a-b)} \sum_{n=1}^{\infty} \frac{(y, z; q)_n x^n q^{an^2+(b-2a)n}(1-q^{2an+b-a})}{(xy, xz; q)_n}\right).$$

*Proof.* In Corollary 1 set $\alpha_0 = 1$ and, for $n > 0$,

$$\alpha_n = q^{an^2+bn} - q^{a(n-1)^2+b(n-1)} = -q^{an^2+(b-2a)n+a-b}(1-q^{2an+b-a}).$$

□

**Corollary 12.**

$$\sum_{n=0}^{\infty} \frac{(1+q^{-2n+3})q^{n^2+6n}}{(q^4; q^4)_n} = \frac{1}{(q^2, q^3; q^5)_\infty (-q^2; q^2)_\infty}. \tag{5.3}$$

$$\sum_{n=0}^{\infty} \frac{(1+q^{-2n+1})q^{n^2+4n}}{(q^4; q^4)_n} = \frac{1}{(q, q^4; q^5)_\infty (-q^2; q^2)_\infty}. \tag{5.4}$$

*Proof.* In (5.2), set $z = 0$, replace $x$ by $x/y$ and let $y \to \infty$ to get

$$\sum_{n=0}^{\infty} \frac{(-x)^n q^{an^2+bn+n(n-1)/2}}{(xq; q)_n} = (1-x) \tag{5.5}$$
$$\times \left(1 - q^{(a-b)} \sum_{n=1}^{\infty} \frac{(-x)^n q^{an^2+(b-2a)n+n(n-1)/2}(1-q^{2an+b-a})}{(x; q)_n}\right).$$

Next, let $a = -1/4$, $b = 1$, replace $q$ by $q^4$ and let $x \to 1$ to get

$$\sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2+2n}}{(q^4; q^4)_n} = -q^{-5} \sum_{n=1}^{\infty} \frac{(-1)^n q^{n^2+4n}(1-q^{-2n+5})}{(q^4; q^4)_{n-1}}.$$

Replace $q$ by $-q$, re-index the right side by replacing $n$ by $n+1$ and (5.3) follows from the following identity of Rogers ([17], page 331):

$$\sum_{n=0}^{\infty} \frac{q^{n^2+2n}}{(q^4;q^4)_n} = \frac{1}{(q^2,q^3;q^5)_\infty(-q^2;q^2)_\infty}.$$

The identity at (5.4) follows similarly, using instead $a=-1/4$, $b=1/2$ in (5.5) and employing another identity of Rogers ([17], page 330):

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q^4;q^4)_n} = \frac{1}{(q,q^4;q^5)_\infty(-q^2;q^2)_\infty}.$$

□

**Corollary 13.**

$$\sum_{n=0}^{\infty} \frac{(b,q^3/b;q)_n q^{n(n+1)/2}}{(q^2;q^2)_{n+1}(q;q)_n} = \frac{(q^4/b,bq;q^2)_\infty}{(q;q)_\infty}. \tag{5.6}$$

$$\sum_{n=0}^{\infty} \frac{(1-q^{2n+1})(-q^3;q^2)_n q^{n^2}}{(q^2;q^2)_n} = \frac{1}{(q^3,q^4,q^5;q^8)_\infty}. \tag{5.7}$$

$$\sum_{n=0}^{\infty} \frac{(1-q^{2n-1})(-q^3;q^2)_n q^{n^2-2n}}{(q^2;q^2)_n} = \frac{1}{(q,q^4,q^7;q^8)_\infty}. \tag{5.8}$$

$$1+\sum_{n=1}^{\infty} \frac{(-q;q)_n q^{(n^2-n)/2}}{(q;q)_{n-1}} = \frac{(-1;q)_\infty(-q^6,-q^{10},q^{16};q^{16})_\infty}{(q^4;q^4)_\infty}. \tag{5.9}$$

$$-1+\sum_{n=1}^{\infty} \frac{(-q;q)_n q^{(n^2-n)/2}}{(q;q)_{n-1}} = q\frac{(-1;q)_\infty(-q^2,-q^{14},q^{16};q^{16})_\infty}{(q^4;q^4)_\infty}. \tag{5.10}$$

*Proof.* In (4.1), let $z=0$, replace $x$ by $x/y$ and let $y\to\infty$ to get

$$\sum_{n=0}^{\infty} \frac{(aq,bq;q)_n(-x)^n q^{n(n-1)/2}}{(qx,abq,q;q)_n} = (1-x)\sum_{n=0}^{\infty} \frac{(a,b;q)_n(-xq)^n q^{n(n-1)/2}}{(x,abq,q;q)_n}.$$

Then set $x=-q$, $a=b/q$ and then use Andrews' $q$-Bailey identity,

$$\sum_{n=0}^{\infty} \frac{(b,q/b;q)_n c^n q^{n(n-1)/2}}{(c;q)_n(q^2;q^2)_n} = \frac{(cq/b,bc;q^2)_\infty}{(c;q)_\infty}$$

with $c=q^2$, to sum the right side. Finally, replace $b$ by $b/q$ and (5.6) follows after a slight manipulation.

For the remaining identities, in (4.1) replace $x$ by $x/y$, let $y\to\infty$ and then set $z=x$ and $b=0$ to get

$$\sum_{n=0}^{\infty} \frac{(1+xq^n)(aq;q)_n(-x)^n q^{n(n-1)/2}}{(q;q)_n} = \sum_{n=0}^{\infty} \frac{(a;q)_n(-x)^n q^{n(n+1)/2}}{(q;q)_n}.$$

For (5.7) and (5.8), replace $q$ by $q^2$, set $a=-q$ and, respectively, $x=-q$ and $x=-1/q$, and use the Göllnitz-Gordon-Slater identities ([12], [13], [22])

$$\sum_{n=0}^{\infty}\frac{q^{n^2+2n}(-q;q^2)_n}{(q^2;q^2)_n}=\frac{1}{(q^3;q^8)_\infty(q^4;q^8)_\infty(q^5;q^8)_\infty},\tag{5.11}$$

$$\sum_{n=0}^{\infty}\frac{q^{n^2}(-q;q^2)_n}{(q^2;q^2)_n}=\frac{1}{(q;q^8)_\infty(q^4;q^8)_\infty(q^7;q^8)_\infty},$$

to sum the respective right sides.

For (5.9), set $a=x=-1$ and use the following identity of Gessel and Stanton ([11], page 196)

$$1+\sum_{n=1}^{\infty}\frac{(-q;q)_{n-1}q^{(n^2+n)/2}}{(q;q)_n}=\frac{(-q;q)_\infty(-q^6,-q^{10},q^{16};q^{16})_\infty}{(q^4;q^4)_\infty}.$$

to sum the resulting right side. The identity at (5.10) follows similarly, again with $a=x=-1$, upon using another identity of Gessel and Stanton ([11], page 196)

$$\sum_{n=0}^{\infty}\frac{(-q;q)_n q^{(n^2+3n)/2}}{(q;q)_{n+1}}=\frac{(-q;q)_\infty(-q^2,-q^{14},q^{16};q^{16})_\infty}{(q^4;q^4)_\infty}.$$

□

**Corollary 14.**

$$\sum_{n=0}^{\infty}\frac{(-q^2;q^2)_{n+1}q^{n^2+2n}}{(q;q)_{2n+3}}=2(-q^2,-q^{14},q^{16};q^{16})_\infty\frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}-\frac{1}{1-q}.\tag{5.12}$$

$$\sum_{n=0}^{\infty}\frac{(-q^2;q^2)_n q^{n^2}}{(q;q)_{2n+1}}=2\frac{(q^2,q^{14},q^{16};q^{16})_\infty(q^{12},q^{20};q^{32})_\infty}{(q;q)_\infty}-1.\tag{5.13}$$

*Proof.* We use (5.1) to prove these identities. First, let $z\to 0$ and replace $q$ with $q^2$ to get

$$\sum_{n=0}^{\infty}\frac{(y;q^2)_n x^n q^{an^2+bn}}{(q^2xy;q^2)_n(-q^{a+b};q^{2a})_n}=\frac{(1-xy)}{(1-x)}\left(1-q^{a-b}\sum_{n=1}^{\infty}\frac{(y;q^2)_n x^n q^{an^2+(b-2a)n}}{(xy;q^2)_n(-q^{a+b};q^{2a})_n}\right).\tag{5.14}$$

For (5.12), set $a=1$, $b=2$, $y=-q^2$, and $x=-1$. Replace $q$ with $-q$, divide both sides by $(1-q)(1-q^2)$ and use Slater's identity **69** to sum the resulting left side:

$$\sum_{n=0}^{\infty}\frac{(-q^2;q^2)_n q^{n^2+2n}}{(q;q)_{2n+2}}=(-q^2,-q^{14},q^{16};q^{16})_\infty\frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}.$$

The result follows after some slight manipulation.

The proof of (5.13) is similar, except we set $a = 1$, $b = 0$, $y = -1$, and $x = -1$, replace $q$ with $-q$, and use Slater's identity **121**:

$$1 + \sum_{n=1}^{\infty} \frac{(-q^2; q^2)_{n-1} q^{n^2}}{(q; q)_{2n}} = \frac{(q^2, q^{14}, q^{16}; q^{16})_\infty (q^{12}, q^{20}; q^{32})_\infty}{(q; q)_\infty}.$$

□

## 6. Representation of False Theta Series as basic Hypergeometric Series

In this section we derive some new representations of the false theta series $\sum_{n=0}^{\infty} (-1)^n q^{n(n+1)/2}$ and $\sum_{n=0}^{\infty} q^{n(3n+1)/2}(1 - q^{2n+1})$, as basic hypergeometric series.

On page 13 of the Lost Notebook [16] (see also [3, page 229]), Ramanujan recorded the following identity (amongst others in a similar vein):

$$\sum_{n=0}^{\infty} \frac{(q; q^2)_n (-1)^n q^{n^2+n}}{(-q; q)_{2n+1}} = \sum_{n=0}^{\infty} (-1)^n q^{n(n+1)/2}. \tag{6.1}$$

On page 37 of the Lost Notebook, he recorded the identities

$$\begin{aligned} \sum_{n=0}^{\infty} q^{n(3n+1)/2}(1 - q^{2n+1}) &= \sum_{n=0}^{\infty} \frac{q^{2n^2+n}}{(-q; q)_{2n+1}} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n+1)/2}}{(-q; q)_n}. \end{aligned} \tag{6.2}$$

The identity that follows from equating the left side to the second right side above also follows as a special case of a more general identity first stated by Rogers [18].

We use these identities in conjunction with (5.14) to prove the following.

**Corollary 15.**

$$1 - \sum_{n=0}^{\infty} \frac{(q; q^2)_n (-1)^n q^{n^2-n}}{(-1; q)_{2n+1}} = \sum_{n=0}^{\infty} (-1)^n q^{n(n+1)/2}. \tag{6.3}$$

$$\frac{2}{1+q} - \sum_{n=0}^{\infty} \frac{q^{2n^2+3n}}{(-q; q)_{2n+1}(1 + q^{2n+3})} = \sum_{n=0}^{\infty} q^{n(3n+1)/2}(1 - q^{2n+1}). \tag{6.4}$$

$$\frac{1}{2} + \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n+1)/2}}{(-1; q)_{n+2}} = \sum_{n=0}^{\infty} q^{n(3n+1)/2}(1 - q^{2n+1}). \tag{6.5}$$

*Proof.* For (6.3), set $a = b = 1$, $y = q$ and $x = -1$ in (5.14). Then divide both sides of the resulting identity by $1 + q$, so that the left side becomes the left side of (6.1). The result follows after re-indexing the resulting sum on the right side, together with a little manipulation.

For (6.4), replace $x$ with $x/y$ in (5.14) and let $y \to \infty$ to get

$$\sum_{n=0}^{\infty} \frac{(-x)^n q^{(a+1)n^2+(b-1)n}}{(q^2x;q^2)_n(-q^{a+b};q^{2a})_n} = (1-x)\left(1 - q^{a-b}\sum_{n=1}^{\infty} \frac{(-x)^n q^{(a+1)n^2+(b-2a-1)n}}{(x;q^2)_n(-q^{a+b};q^{2a})_n}\right). \tag{6.6}$$

Then set $a = 1$, $b = 2$, $x = -1$, and divide both sides by $1 + q$ so that the left side becomes the first right side of (6.2). The result again follows, upon re-indexing the sum on the right side.

To get (6.3), set $y = 0$ in (5.14), then $a = b = 1/2$ and $x = -1$, so the left side becomes the second right side in (6.2). The result likewise follows after re-indexing the resulting sum on the right side. □

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