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Ramanujan and the Regular Continued Fraction Expansion of Real Numbers

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RAMANUJAN AND THE REGULAR CONTINUED FRACTION EXPANSION OF REAL NUMBERS

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Abstract. In some recent papers, the authors considered regular continued fractions of the form

$$[a_0; \underbrace{a, \cdots, a}_{m}, \underbrace{a^2, \cdots, a^2}_{m}, \underbrace{a^3, \cdots, a^3}_{m}, \cdots],$$

where $a_0 \geq 0$, $a \geq 2$ and $m \geq 1$ are integers. The limits of such continued fractions, for general a and in the cases $m = 1$ and $m = 2$, were given as ratios of certain infinite series.

However, these formulae can be derived from known facts about two continued fractions of Ramanujan. Motivated by these observations, we give alternative proofs of the results of the previous authors for the cases $m = 1$ and $m = 2$ and also use known results about other q-continued fractions investigated by Ramanujan to derive the limits of other infinite families of regular continued fractions.

1. Introduction

It is an interesting problem to try to find irrational numbers whose regular continued fraction expansion contains predictable patterns and which can be expressed in some other form.

The most familiar class of such numbers comprises the quadratic irrationalities, $\alpha = p + q\sqrt{D}$, where p and q are rational, $q \neq 0$ and D is a non-square positive integer. Such numbers have a regular continued fraction expansion which is ultimately periodic:

$$\alpha = [a_0; a_1, \cdots, a_k, \overline{b_1, \cdots, b_n}\,].$$

Another class consists of the *Hurwitzian* continued fractions of the form

$$[a_0; a_1, \cdots, a_k, f_1(1), \cdots, f_n(1), f_1(2), \cdots, f_n(2), \cdots] \\ =: [a_0; a_1, \cdots, a_k, \overline{f_1(m), \cdots, f_n(m)}\,]_{m=1}^{\infty}.$$

Here the $f_i(x)$ are polynomials with rational coefficients taking only positive integral values for integral $x \geq 1$ and at least one is non-constant. The closed form for Hurwitzian continued fractions is not known in general. This class contains numbers like

$$e^{2/(2n+1)} = [1; n, 12n+6, 5n+2, \\ \overline{1, 1, (6m+1)n+3m,\ (24m+12)n+12m+6,\ (6m+5)n+3m+2}\,]_{m=1}^{\infty}.$$

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A third class is due to D.H. Lehmer [7], who found closed forms for the numbers represented by regular continued fractions whose partial quotients were either terms in an arithmetic progression,

$$[0; a, a + c, a + 2c, a + 3c, \cdots],$$

or terms in two interlaced arithmetic progressions,

$$[0; a, b, a + c, b + d, a + 2c, b + 2d, \cdots].$$

An example that Lehmer gave of the former type was the following:

$$[1; 2, 3, 4, 5, \cdots] = \frac{\sum_{m=0}^{\infty} \frac{1}{(m!)^2}}{\sum_{m=0}^{\infty} \frac{1}{m!(m+1)!}}$$

Tasoev [11], [12] proposed a new type of continued fraction of the form

$$[a_0; \underbrace{a, \cdots, a}_{m}, \underbrace{a^2, \cdots, a^2}_{m}, \underbrace{a^3, \cdots, a^3}_{m}, \cdots], \tag{1.1}$$

where $a_0 \geq 0$, $a \geq 2$ and $m \geq 1$ are integers. This type was further investigated by Komatsu [5], who derived a closed form for the general case ($m \geq 1$, arbitrary). For the special cases $m = 1$ and $m = 2$, he derived the following expressions. As usual, the empty product denotes 1.

$$[0, \overline{a^k}]_{k=1}^{\infty} := [0; a, a^2, a^3, a^4, \cdots] = \frac{\sum_{s=0}^{\infty} a^{-(s+1)^2} \prod_{i=1}^{s} (a^{2i} - 1)^{-1}}{\sum_{s=0}^{\infty} a^{-s^2} \prod_{i=1}^{s} (a^{2i} - 1)^{-1}}. \tag{1.2}$$

$$[0, \overline{a^k, a^k}]_{k=1}^{\infty} := [0; a, a, a^2, a^2, \cdots] = \frac{\sum_{s=0}^{\infty} a^{-(s+1)(s+2)/2} \prod_{i=1}^{s} (a^i - 1)^{-1}}{\sum_{s=0}^{\infty} a^{-s(s+1)/2} \prod_{i=1}^{s} (a^i - 1)^{-1}}. \tag{1.3}$$

In [6] he generalized these results and gave similar expressions for continued fractions like $[0, \overline{ua^k}]$ and $[0, ua - 1, \overline{1, ua^{k+1} - 2}]$, with $a > 1$ an integer and u rational such that $ua \in \mathbb{Z}^+$.

Komatsu manipulated certain infinite series to derive his results. In this present paper we use known facts about certain q-continued fractions studied by Ramanujan to give alternative, perhaps simpler, derivations of some of Komatsu's results.

Motivated by this connection between families of regular continued fraction expansions and q-continued fractions, we investigated other q-continued fractions studied by Ramanujan and were able to derive other infinite families of regular continued fraction expansions which can be summed in a closed form. Here are some examples of our results (proofs are found throughout the paper):

Example 1. *If $c > 1$, $a > 1$ are integers and d is rational such that $da > 1$, then*

$$[c - 1; \overline{1, da^k - 1, ca^k - 1}]_{k=1}^{\infty} = \frac{\sum_{n=0}^{\infty} \dfrac{(-1)^n}{c^{n-1} d^n a^{n^2} (-1/a; -1/a)_n}}{\sum_{n=0}^{\infty} \dfrac{1}{(cd)^n a^{n^2+n} (-1/a; -1/a)_n}}.$$

Example 2. *Let $a > 1$ be an integer. Then*

$$[1; \overline{a^{2k-1} - 1, a^{2k} + 1}]_{k=1}^{\infty} = \frac{(-1/a; -1/a^3)_{\infty}}{(1/a^2; -1/a^3)_{\infty}}.$$

Example 3. *Define*

$$F(c,d,q) := \sum_{n=0}^{\infty} \frac{(-1)^n c^n q^{n(n+1)/2}}{(q;q)_n (cq/d;q)_n}$$

and let $\omega = e^{2\pi i/3}$. If $a > 1$ is an integer and c is a rational such that a/c is an integer, $a/c > 2$, then

$$\left[0; \frac{a}{c} - 2, \overline{1, \frac{a^{k+1}}{c} - 3}\right]_{k=1}^{\infty} = \frac{c/a\, F(-c\omega/a, \omega^2, 1/a)}{(1 + c\omega^2/a) F(-c\omega, \omega^2, 1/a)}.$$

Example 4. *For r, s and $q \in \mathbb{C}$ with $|q| < 1$, define*

$$\phi(r,s,q) = \sum_{n=0}^{\infty} \frac{q^{(n^2+n)/2} r^n}{(q;q)_n (-sq;q)_n}.$$

Let m and n be positive integers and let d be rational such that $dn \in \mathbb{Z}^+$ and $dmn > 1$. If $n > 2$ and $m > 1$ then

$$[0, \overline{1, d^{2k-2} n^{2k-1} - 2, 1, m^{2k-1} - 1, d^{2k-1} n^{2k}, m^{2k} - 1}]_{k=1}^{\infty}$$
$$= 1 + \frac{\phi(dm, d, -1/(dmn))}{\phi(-1/n, d, -1/(dmn))}.$$

Curiously, it seems that Ramanujan was not particularly interested in the regular continued fraction expansion of real numbers, or with his flair for continued fractions he would almost certainly have derived our results himself. Perhaps he typified a gap that may exist between people who study q-continued fractions and those who are interested in the regular continued fraction expansion of real numbers?

2. Some Continued Fractions of Ramanujan

In Chapter 16 of the Second Notebook, Ramanujan gave the following corollary to Entry 15 ([2], page 30):

Corollary 1. *If $|q| < 1$, then*

$$G(b,q) := 1 + \frac{bq}{1} + \frac{bq^2}{1} + \frac{bq^3}{1} + \cdots = \frac{\sum_{n=0}^{\infty} \frac{b^n q^{n^2}}{(q;q)_n}}{\sum_{n=0}^{\infty} \frac{b^n q^{n^2+n}}{(q;q)_n}}. \tag{2.1}$$

We have restated this corollary in a form that is more convenient for our purposes. If we set $b = 1$ then the well-known expression for the Rogers-Ramanujan continued fraction follows. We also recall the famous Rogers-Ramanujan identities [10]:

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} = \prod_{j=0}^{\infty} \frac{1}{(1-q^{5j+1})(1-q^{5j+4})} = \frac{1}{(q;q^5)_\infty (q^4;q^5)_\infty},$$

$$\sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q;q)_n} = \prod_{j=0}^{\infty} \frac{1}{(1-q^{5j+2})(1-q^{5j+3})} = \frac{1}{(q^2;q^5)_\infty (q^3;q^5)_\infty},$$

where

$$(c;q)_0 := 1, \quad (c;q)_n = \prod_{j=0}^{n-1}(1-cq^j), \quad (c;q)_\infty = \lim_{n\to\infty}(c;q)_n, \quad |q|<1.$$

If we combine these identities we have that

$$K(q) := 1 + \frac{q}{1} + \frac{q^2}{1} + \frac{q^3}{1} + \cdots = \frac{\sum_{n=0}^{\infty}\frac{q^{n^2}}{(q;q)_n}}{\sum_{n=0}^{\infty}\frac{q^{n^2+n}}{(q;q)_n}} = \frac{(q^2;q^5)_\infty(q^3;q^5)_\infty}{(q;q^5)_\infty(q^4;q^5)_\infty}. \tag{2.2}$$

If we transform $K(q)$ so that

$$K(q) := 1 + \frac{1}{1/q} + \frac{1}{1/q} + \frac{1}{1/q^2} + \frac{1}{1/q^2} + \frac{1}{1/q^3} + \frac{1}{1/q^3} + \cdots$$

and set $q = 1/a$, where $a \geq 2$ is a positive integer, we have that

$$\begin{aligned} 1 + [0; a, a, a^2, a^2, \cdots] &= \frac{\sum_{n=0}^{\infty}\frac{1}{a^{n^2}(1/a;1/a)_n}}{\sum_{n=0}^{\infty}\frac{1}{a^{n^2+n}(1/a;1/a)_n}} \\ &= \frac{(1/a^2;1/a^5)_\infty(1/a^3;1/a^5)_\infty}{(1/a;1/a^5)_\infty(1/a^4;1/a^5)_\infty}. \end{aligned}$$

The first equation, after some elementary manipulation, gives the identity at (1.3).

We will use the identity at (2.1) to prove the following theorem.

Theorem 1. *Let $a \geq 2$ be a positive integer and suppose c and d are rationals such that ca, $da \in \mathbb{Z}^+$. Then*

$$[0; \overline{da^k, ca^k}]_{k=1}^{\infty} = \frac{\sum_{n=0}^{\infty}\frac{1}{c^n d^{n+1} a^{(n+1)^2}(1/a;1/a)_n}}{\sum_{n=0}^{\infty}\frac{1}{(cd)^n a^{n^2+n}(1/a;1/a)_n}}. \tag{2.3}$$

$$[0; \overline{ca^k}]_{k=1}^{\infty} = \frac{\sum_{n=0}^{\infty}\frac{1}{c^{2n+1} a^{2n^2+3n+1}(1/a^2;1/a^2)_n}}{\sum_{n=0}^{\infty}\frac{1}{c^{2n} a^{2n^2+n}(1/a^2;1/a^2)_n}}. \tag{2.4}$$

If $ca > 1$, then

$$[0; ca-1, \overline{1, ca^{k+1}-2}]_{k=1}^{\infty} = \frac{\sum_{n=0}^{\infty}\frac{(-1)^n}{c^{2n+1} a^{2n^2+3n+1}(1/a^2;1/a^2)_n}}{\sum_{n=0}^{\infty}\frac{(-1)^n}{c^{2n} a^{2n^2+n}(1/a^2;1/a^2)_n}}. \tag{2.5}$$

If $da > 1$ and $ca > 2$, then

$$[0; da-1, 1, ca-2, \overline{1, da^{k+1}-2, 1, ca^{k+1}-2}\,]_{k=1}^{\infty} = \frac{\sum_{n=0}^{\infty} \frac{(-1)^n}{c^n d^{n+1} a^{(n+1)^2}(1/a;1/a)_n}}{\sum_{n=0}^{\infty} \frac{(-1)^n}{(cd)^n a^{n^2+n}(1/a;1/a)_n}}. \tag{2.6}$$

If $c > 1$ is an integer and $da > 1$, then

$$[c-1; \overline{1, da^k-1, ca^k-1}\,]_{k=1}^{\infty} = \frac{\sum_{n=0}^{\infty} \frac{(-1)^n}{c^{n-1} d^n a^{n^2}(-1/a;-1/a)_n}}{\sum_{n=0}^{\infty} \frac{1}{(cd)^n a^{n^2+n}(-1/a;-1/a)_n}}. \tag{2.7}$$

Proof. On the left side of Equation 2.1, let $b = 1/(c\,d)$, so that $G(b, q)$ can be written as

$$G(b,q) = 1 + \frac{1}{c}\left(\frac{1}{d/q +}\,\frac{1}{c/q +}\,\frac{1}{d/q^2 +}\,\frac{1}{c/q^2 +}\cdots\right).$$

If we now let $q = 1/a$ and c and d be rationals such that ca, $da \in \mathbb{Z}^+$ then

$$1 + \frac{1}{c}[0; \overline{da^k, ca^k}\,]_{k=1}^{\infty} = \frac{\sum_{n=0}^{\infty} \frac{1}{(cd)^n a^{n^2}(1/a;1/a)_n}}{\sum_{n=0}^{\infty} \frac{1}{(cd)^n a^{n^2+n}(1/a;1/a)_n}}. \tag{2.8}$$

After some manipulation, we have the identity at (2.3).

Remark: Komatsu also has this result in [6] as a corollary to a more general result about continued fractions of the form $[0; \overline{da^k, cb^k}\,]_{k=1}^{\infty}$. If we set $c = d = 1$, some further simplification gives the result from his paper [5] at (1.3).

If we replace a by a^2 and d by c/a in Equation 2.3, we have the identity at (2.4). This is Theorem 1 from [6] and setting $c = 1$ gives the identity at 1.2 from [5].

As shown in [13], negatives and zeroes can easily be removed from regular continued fraction expansions. Indeed, it is easy to check that $[m, n, 0, p, \alpha] = [m, n+p, \alpha]$ and $[m, -n, \alpha] = [m-1, 1, n-1, -\alpha]$. Hence it is possible to allow the parameters a, c and d in Equations 2.8, 2.3 and 2.4 to take negative values. Thus, if we replace c by $-c$ and a by $-a$ in (2.4), for example, and repeatedly apply the second of the above conditions, we have that

$$\begin{aligned}
&[0; ca, -ca^2, ca^3, -ca^4, ca^5, -ca^6, \cdots]\\
&\quad = [0; ca-1, 1, ca^2-1, -ca^3, ca^4, -ca^5, ca^6, \cdots]\\
&\quad = [0; ca-1, 1, ca^2-2, 1, ca^3-1, -ca^4, ca^5, -ca^6, \cdots]\\
&\quad = [0; ca-1, 1, ca^2-2, 1, ca^3-2, 1, ca^4-1, -ca^5, ca^6, \cdots]\\
&\quad = [0; ca-1, 1, ca^2-2, 1, ca^3-2, 1, ca^4-2, 1, ca^5-1, -ca^6, \cdots] \text{ etc.}
\end{aligned}$$

Finally, we have, for $ca > 1$, the identity at (2.5)

This is Theorem 2 from [6]. Likewise, if c is replaced by $-c$ in 2.3 and the resulting continued fraction similarly manipulated, one gets (2.6). This is Theorem 4 from [6]. One that Komatsu missed is (2.7). This is derived from Equation 2.8 by multiplying across by c, replacing a by $-a$ and similarly manipulating the continued fraction to remove the negatives. $\square$

We digress slightly before proving our next result. We introduce some notation from [8] (page 83). We call $d_0 + K_{n=1}^{\infty} c_n/d_n$ a *canonical contraction* of $b_0 + K_{n=1}^{\infty} a_n/b_n$ if

$$C_k = A_{n_k}, \qquad D_k = B_{n_k} \qquad \text{for } k = 0, 1, 2, 3, \dots,$$

where C_n, D_n, A_n and B_n are canonical numerators and denominators of $d_0 + K_{n=1}^{\infty} c_n/d_n$ and $b_0 + K_{n=1}^{\infty} a_n/b_n$ respectively.

From [8] (page 85)we also have the following theorem.

Theorem 2. *The canonical contraction of $b_0 + K_{n=1}^{\infty} a_n/b_n$ with $C_0 = A_1/B_1$*

$$C_k = A_{2k+1} \qquad D_k = B_{2k+1} \qquad \textit{for } k = 1, 2, 3, \dots,$$

exists if and only if $b_{2k+1} \neq 0 for K = 0, 1, 2, 3, \dots,$ and in this case is given by

$$\frac{b_0 b_1 + a_1}{b_1} - \frac{a_1 a_2 b_3/b_1}{b_1(a_3 + b_2 b_3) + a_2 b_3} - \frac{a_3 a_4 b_5 b_1/b_3}{a_5 + b_4 b_5 + a_4 b_5/b_3} - \frac{a_5 a_6 b_7/b_5}{a_7 + b_6 b_7 + a_6 b_7/b_5} - \frac{a_7 a_8 b_9/b_7}{a_9 + b_8 b_9 + a_8 b_9/b_7} + \cdots. \tag{2.9}$$

The continued fraction (2.9) is called the *odd* part of $b_0 + K_{n=1}^{\infty} a_n/b_n$. The following corollary follows easily from Theorem 2.

Corollary 2. *The odd part of the continued fraction*

$$\frac{c_1}{1} - \frac{c_2}{1} + \frac{c_2}{1} - \frac{c_3}{1} + \frac{c_3}{1} - \frac{c_4}{1} + \frac{c_4}{1} - \cdots$$

is

$$c_1 + \frac{c_1 c_2}{1} + \frac{c_2 c_3}{1} + \frac{c_3 c_4}{1} + \cdots.$$

We will also make use of Worpitzky's Theorem (see [8], pp. 35–36).

Theorem 3. *(Worpitzky) Let the continued fraction $K_{n=1}^{\infty} a_n/1$ be such that $|a_n| \leq 1/4$ for $n \geq 1$. Then$K_{n=1}^{\infty} a_n/1$ converges. All approximants of the continued fraction lie in the disc $|w| < 1/2$ and the value of the continued fraction is in the disk $|w| \leq 1/2$.*

We are now ready to prove Theorem 4

Theorem 4. *Let m, r, s and $a > 1$ be positive integers such that $sa/(rm^2)$ is an integer. If $m > 1$ and $sa/(rm^2) > 1$, then*

$$\left[0; \overline{m-1, 1, \frac{sa^k}{rm^2} - 1, m-1, 1, a^k - 1}\right]_{k=1}^{\infty} = \frac{1}{m} + \frac{\sum_{n=0}^{\infty} \frac{(r/s)^{n+1}}{a^{(n+1)^2}(1/a; 1/a)_n}}{\sum_{n=0}^{\infty} \frac{(r/s)^n}{a^{n^2+n}(1/a; 1/a)_n}}. \tag{2.10}$$

If $m > 2$ and $sa/(rm^2) > 1$, then

$$\left[1; \overline{m, \frac{sa^k}{rm^2} - 1, 1, m-2, 1, a^k - 1}\right]_{k=1}^{\infty} = \frac{1}{m} + \frac{\sum_{n=0}^{\infty} \frac{(-r/s)^n}{a^{n^2}(1/a; 1/a)_n}}{\sum_{n=0}^{\infty} \frac{(-r/s)^n}{a^{n^2+n}(1/a; 1/a)_n}}. \tag{2.11}$$

If $m>2$ and $sa/(rm^2)>1$, then

$$\left[0;\overline{1,m-2,1,\frac{sa^k}{rm^2}-1,m,a^k-1}\right]_{k=1}^{\infty}=\frac{-1}{m}+\frac{\sum_{n=0}^{\infty}\frac{(-r/s)^n}{a^{n^2}(1/a;1/a)_n}}{\sum_{n=0}^{\infty}\frac{(-r/s)^n}{a^{n^2+n}(1/a;1/a)_n}}. \tag{2.12}$$

If $m>2$ and $sa/(rm^2)>1$, then

(2.13)

$$\left[1;\overline{m,\frac{sa^{2k-1}}{rm^2}-1,1,m-1,a^{2k-1}-1,1,m-2,1,\frac{sa^{2k}}{rm^2}-1,m-1,1,a^{2k}-1}\right]_{k=1}^{\infty}$$
$$=\frac{1}{m}+\frac{\sum_{n=0}^{\infty}\frac{(-r/s)^n}{a^{n^2}(-1/a;-1/a)_n}}{\sum_{n=0}^{\infty}\frac{(r/s)^n}{a^{n^2+n}(-1/a;-1/a)_n}}.$$

Proof. Let c_1 be arbitrary and apply Corollary 2 to $G(b,q)$ from Corollary 1, so that

$$\begin{aligned} G(b,q) &= 1-c_1+\frac{c_1}{1}-\frac{bq/c_1}{1}+\frac{bq/c_1}{1}-\frac{qc_1}{1}+\frac{qc_1}{1} \\ &\qquad -\frac{bq^2/c_1}{1}+\frac{bq^2/c_1}{1}-\frac{q^2c_1}{1}+\frac{q^2c_1}{1}-\cdots \\ &= 1-c_1+\frac{1}{1/c_1}-\frac{1}{c_1^2/(bq)}+\frac{1}{1/c_1}-\frac{1}{1/q}+\frac{1}{1/c_1} \\ &\qquad -\frac{1}{c_1^2/(bq^2)}+\frac{1}{1/c_1}-\frac{1}{1/q^2}+\frac{1}{1/c_1}- \\ &\qquad \cdots \\ &\qquad -\frac{1}{c_1^2/(bq^n)}+\frac{1}{1/c_1}-\frac{1}{1/q^n}+\frac{1}{1/c_1}-\cdots. \end{aligned} \tag{2.14}$$

(2.15)

The first equality is valid since the continued fraction on the left converges by Worpitzky's Theorem and hence equals its odd part, which is $G(b,q)$, by Corollary 2.

If we now let $c_1=1/m$, where m is an integer, $b=r/s$, where r and s are integers, and $q=1/a$, where a is an integer such that $sa/(rm^2)$ is an integer, we have, after a little manipulation to bring all the negative signs into the denominators, that

$$\begin{aligned} G(r/s,1/a) &= 1-\frac{1}{m}+[0;\overline{m,-sa^k/(rm^2),-m,a^k}\,]_{k=1}^{\infty} \\ &= \frac{\sum_{n=0}^{\infty}\frac{(r/s)^n}{a^{n^2}(1/a;1/a)_n}}{\sum_{n=0}^{\infty}\frac{(r/s)^n}{a^{n^2+n}(1/a;1/a)_n}}. \end{aligned} \tag{2.16}$$

One can now make various choices for the signs of the parameters m, s, r and a and remove the negative signs as before to produce regular continued fraction expansions. If we choose all parameters to be positive, then clearing negatives and a little manipulation on the right side of Equation 2.16 gives (2.10).

Remark: We point out something that is a little curious. Suppose we choose $r = s = 1$ in (2.10) and suppose further that $m_1^2|a$ and $m_2^2|a$. Then

$$\left[0; \overline{m_1 - 1, 1, \frac{a^k}{m_1^2}, m_1 - 1, 1, a^k - 1}\right]_{k=1}^{\infty} = \left[0; \overline{m_2 - 1, 1, \frac{a^k}{m_2^2}, m_2 - 1, 1, a^k - 1}\right]_{k=1}^{\infty} + \frac{1}{m_1} - \frac{1}{m_2}. \tag{2.17}$$

In other words, different values of the parameter m produces continued fractions whose values differ by a rational number. This is clear since the only explicit appearance of m on the right of (2.10) is in the fraction $1/m$.

If we replace r by $-r$ in (2.16), then after clearing negatives we get (2.11).
Remark: If we let $m = 2$ in (2.11), then the resulting zeroes can be removed as described in the proof of Theorem 1, to give that, if $4r|sa$ and $sa/4r > 1$, then

$$\left[1; \overline{2, \frac{sa^k}{4r} - 1, 2, a^k - 1}\right]_{k=1}^{\infty} = \frac{1}{2} + \frac{\sum_{n=0}^{\infty} \frac{(-r/s)^n}{a^{n^2}(1/a;1/a)_n}}{\sum_{n=0}^{\infty} \frac{(-r/s)^n}{a^{n^2+n}(1/a;1/a)_n}}. \tag{2.18}$$

Similar transformations of some of the other continued fractions in the paper are possible. However, we generally ignore this.

If we replace m by $-m$ and s by $-s$ in (2.16), then clearing negatives gives (2.12). Finally, replacing a by $-a$ (2.16) and clearing negatives gives (2.13).

Other choices of signs will lead to different regular continued fraction expansions at (2.16). $\square$

On page 290 of his second notebook, Ramanujan recorded the following continued fraction identity.

Proposition 1. *(see [3], page 46, Entry 19) For* $|q| < 1$,

$$\frac{(q^2;q^3)_\infty}{(q;q^3)_\infty} = \frac{1}{1} - \frac{q}{1+q} - \frac{q^3}{1+q^2} - \frac{q^5}{1+q^3} - \cdots. \tag{2.19}$$

A proof of this identity can be found in [1]. We will use (2.19) to prove the following theorem.

Theorem 5. *Let* $a > 1$ *be an integer. Then*

$$[1; \overline{a^{2k-1} - 1, a^{2k} + 1}]_{k=1}^{\infty} = \frac{(-1/a; -1/a^3)_\infty}{(1/a^2; -1/a^3)_\infty}. \tag{2.20}$$

$$[1; \overline{a^k - 1, 1}]_{k=1}^{\infty} = \frac{(1/a^2; 1/a^3)_\infty}{(1/a; 1/a^3)_\infty}. \tag{2.21}$$

Proof. Replace q by $-q$ in (2.19) so that

$$\begin{aligned}\frac{(q^2;-q^3)_\infty}{(-q;-q^3)_\infty} &= \frac{1}{1}+\frac{q}{1-q}+\frac{q^3}{1+q^2}+\frac{q^5}{1-q^3}-\cdots \\ &= \frac{1}{1}+\frac{1}{1/q-1}+\frac{1}{1/q^2+1}+\frac{1}{1/q^3-1}-\cdots \\ &= [0;1,1/q-1,1/q^2+1,1/q^3-1,1/q^4+1,\cdots]\end{aligned} \tag{2.22}$$

If q is now replaced by $1/a$, where $a>1$ is an integer, and both sides of (2.22) are inverted, then (2.20) follows. If q is replaced by $-1/a$ and the resulting negatives are removed from the continued fraction, then (2.21) follows. $\square$

In [1], the authors generalize the identity at (2.19) as follows.

Proposition 2. *Define*

$$F(c,d,q) := \sum_{n=0}^{\infty} \frac{(-1)^n c^n q^{n(n+1)/2}}{(q;q)_n (cq/d;q)_n}$$

and let $\omega = e^{2\pi i/3}$. *Then*

$$\frac{c\,q\,F(c\omega q,\omega^2,q)}{(1-c\omega^2 q)F(c\omega,\omega^2,q)} = \frac{cq}{1+cq}-\frac{c^2q^3}{1+cq^2}-\frac{c^2q^5}{1+cq^3}-\cdots. \tag{2.23}$$

We have changed the statement of their identity slightly to better suit our purposes and to avoid conflict with our already existing notation. We use the identity above to prove the following theorem.

Theorem 6. *Let a be a positive integer and let c be rational such that $a/c \in \mathbb{Z}^+$. If $a/c>1$, then*

$$\left[0;\overline{\frac{a^{2k-1}}{c}-1,\frac{a^{2k}}{c}+1}\right]_{k=1}^{\infty} = \frac{c/a\,F(-c\omega/a,\omega^2,-1/a)}{(1+c\omega^2/a)F(c\omega,\omega^2,-1/a)}. \tag{2.24}$$

$$\left[0;\overline{\frac{a^{2k-1}}{c}+1,\frac{a^{2k}}{c}-1}\right]_{k=1}^{\infty} = \frac{c/a\,F(c\omega/a,\omega^2,-1/a)}{(1-c\omega^2/a)F(-c\omega,\omega^2,-1/a)}. \tag{2.25}$$

If $a/c>2$, then

$$\left[0;\frac{a}{c}-2,\overline{1,\frac{a^{k+1}}{c}-3}\right]_{k=1}^{\infty} = \frac{c/a\,F(-c\omega/a,\omega^2,1/a)}{(1+c\omega^2/a)F(-c\omega,\omega^2,1/a)}. \tag{2.26}$$

$$\left[0;\overline{1,\frac{a^{k}}{c}-1}\right]_{k=1}^{\infty} = 1-\frac{c/a\,F(c\omega/a,\omega^2,1/a)}{(1-c\omega^2/a)F(c\omega,\omega^2,1/a)}. \tag{2.27}$$

Proof. Replace q by $-q$ in (2.23) and cancel a negative sign from both sides to get that

$$\begin{aligned}\frac{c\,q\,F(-c\omega q,\omega^2,-q)}{(1+c\omega^2 q)F(c\omega,\omega^2,-q)} &= \frac{cq}{1-cq}+\frac{c^2q^3}{1+cq^2}+\frac{c^2q^5}{1-cq^3}+\cdots \\ &= \frac{1}{1/(cq)-1}+\frac{1}{1/(cq^2)+1}+\frac{1}{1/(cq^3)-1}+\cdots \\ &= [0;\overline{1/(cq^{2k-1})-1,1/(cq^{2k})+1}]_{k=1}^{\infty}.\end{aligned} \tag{2.28}$$

If we let $q = 1/a$, where $a > 1$, $a \in \mathbb{Z}^+$ and $a/c > 1$, $a/c \in \mathbb{Z}^+$, then (2.24) follows. If c is replaced by $-c$ in (2.24) and both sides are multiplied by -1, then (2.25) follows.

If c is replaced by $-c$ and a is replaced by $-a$ in (2.24) and the resulting negatives in the continued fraction are cleared, then (2.26) follows.

Equation (2.27) follows from (2.24) upon replacing a by $-a$, removing the resulting negatives from the continued fraction and moving the initial -1 that appears at the beginning of the continued fraction to the other side of the equation. □

In proving Proposition 2 the authors in make use of the following property of the function $F(a, b, q)$:

$$\frac{(1-aq/b)F(a,b,q)}{F(aq,b,q)} = (1-aq-aq/b) - \frac{a^2q^3/b}{\frac{(1-aq^2/b)F(aq,b,q)}{F(aq^2,b,q)}}.$$

Upon iteration the identity above produces the following continued fraction:

$$\frac{(1-aq/b)F(a,b,q)}{F(aq,b,q)} = (1-aq-aq/b) - \frac{a^2q^3/b}{1-aq^2-aq^2/b} - \frac{a^2q^5/b}{1-aq^3-aq^3/b} - \frac{a^2q^7/b}{1-aq^4-aq^4/b} - \cdots. \tag{2.29}$$

The authors in [1] did not consider this continued fraction in its full generality, so did not prove that the right side converged and equalled the left side, but this follows easily using an argument similar to the one they used for the special case $a = \omega$, $b = \omega^2$. Namely, the continued fraction at (2.29) is equivalent to a continued fraction of the form $c_0 + K_{n=1}^{\infty} c_n/1$, where, for $n \geq 2$,

$$c_n = \frac{a^2q^{2n+1}}{(1-aq^n-aq^n/b)(1-aq^{n+1}-aq^{n+1}/b)}.$$

Worpitzky's theorem gives that the continued fraction converges. Secondly,

$$\frac{(1-aq^n/b)F(aq^{n-1},b,q)}{F(aq^n,b,q)} = 1 + O(q^n)$$

as $n \to \infty$ and this is sufficient to show that the right side of (2.29) converges to the left side. We now use this continued fraction to prove the following theorem.

Theorem 7. *Let $n > 1$ be an integer and m a positive rational such that $m\,n$ is integral. If $m\,n > 4$, then*

$$[mn-3, \overline{1, mn^k-4}]_{k=2}^{\infty} = (mn-1)\frac{F(1/m,1,1/n)}{F(1/(mn),1,1/n)}. \tag{2.30}$$

If $m\,n > 2$, then

$$[mn-2, \overline{mn^{2k}+2, mn^{2k+1}-2}]_{k=1}^{\infty} = (mn-1)\frac{F(-1/m,1,-1/n)}{F(1/(mn),1,-1/n)}. \tag{2.31}$$

$$[mn+1, \overline{1, mn^k}]_{k=2}^{\infty} = (mn+1)\frac{F(-1/m,1,1/n)}{F(-1/(mn),1,1/n)}. \tag{2.32}$$

If $m\,n^2 > 2$, then

$$[mn+2, \overline{mn^{2k}-2, mn^{2k+1}+2}]_{k=1}^{\infty} = (mn-1)\frac{F(1/m,1,-1/n)}{F(-1/(mn),1,-1/n)}. \tag{2.33}$$

Proof. In (2.29) above, let $b = 1$, divide both sides by $a\,q$ and then apply a series of similarity transformations to bring all terms into the denominator. This gives that

$$\frac{(1/aq-1)F(a,1,q)}{F(aq,1,q)} = \frac{1}{aq} - 2 - \frac{1}{1/aq^2-2}\,-\,\frac{1}{1/aq^3-2}\,-\,\frac{1}{1/aq^4-2}\,-\cdots$$
$$= \frac{1}{aq} - 2 + \frac{1}{-1/aq^2+2}\,+\,\frac{1}{1/aq^3-2}\,+\,\frac{1}{-1/aq^4+2}\,+\,\frac{1}{1/aq^5-2}\,+\cdots.$$

Now let $a = 1/n$ and $q = 1/n$ and we have that

$$(mn-1)\frac{F(1/m,1,1/n)}{F(1/(mn),1,1/n)} = [mn-2, \overline{-mn^{2k}+2, mn^{2k+1}-2}]_{k=1}^{\infty}. \tag{2.34}$$

Upon removing the negatives from the left side of (2.34) we get (2.30).

Equation (2.31) follows from (2.34) upon replacing m by $-m$, n by $-n$ and likewise removing the negatives.

Equation (2.32) follows similarly upon replacing n by $-n$ and (2.33) follows upon replacing m by $-m$ (and in each case removing the negatives).

Remark: The only other value of b which leads to regular continued fraction expansions is $b = -1$. However, this produce a variant of the generalized Rogers-Ramanujan continued fraction and, in light of Theorem1, leads to no new types of regular continued fraction expansions.

□

On page 374 of his second notebook, Ramanujan recorded the following continued fraction identity (see [3], page 45, Entry 17).

Proposition 3. *Let c, d and q be complex numbers with $|q| < 1$. Define*

$$\phi(c,d,q) = \sum_{n=0}^{\infty} \frac{q^{(n^2+n)/2}c^n}{(q;q)_n(-dq:q)_n}.$$

Then

$$\frac{\phi(c,d,q)}{\phi(cq,d,q)} = 1 + \frac{cq}{1}\,+\,\frac{dq}{1}\,+\,\frac{cq^2}{1}\,+\,\frac{dq^2}{1}\,+\,\frac{cq^3}{1}\,+\,\frac{dq^3}{1}\,+\cdots. \tag{2.35}$$

Once again we have changed the statement of this identity slightly to better suit our purposes. Equation 2.35 can be rewritten as

$$\frac{\phi(c,d,q)}{\phi(cq,d,q)} = 1 + \frac{1}{1/(cq)}\,+\,\frac{1}{c/d}\,+\,\frac{1}{d/(cq)^2}\,+\,\frac{1}{(c/d)^2}\,+\,\frac{1}{d^2/(cq)^3}\,+\cdots \tag{2.36}$$
$$= [1, \overline{d^{k-1}/(cq)^k, (c/d)^k}]_{k=1}^{\infty}.$$

If we let $c = dm$ and $q = 1/(dmn)$, this identity becomes

$$\frac{\phi(dm,d,1/(dmn))}{\phi(1/n,d,1/(dmn))} = [1, \overline{d^{k-1}n^k, m^k}]_{k=1}^{\infty}. \tag{2.37}$$

From this identity we can deduce the following theorem.

Theorem 8. *For r, s and $q \in \mathbb{C}$ with $|q| < 1$, define*

$$\phi(r,s,q) = \sum_{n=0}^{\infty} \frac{q^{(n^2+n)/2}r^n}{(q;q)_n(-sq;q)_n}.$$

Let m and n be positive integers and let d be rational such that $dn \in \mathbb{Z}^+$ and $dmn > 1$. Then

$$[1, \overline{d^{k-1}n^k, m^k}]_{k=1}^{\infty} = \frac{\phi(dm, d, 1/(dmn))}{\phi(1/n, d, 1/(dmn))}. \tag{2.38}$$

If $n > 2$ and $m > 1$ then

$$[0, \overline{1, d^{2k-2}n^{2k-1} - 2, 1, m^{2k-1} - 1, d^{2k-1}n^{2k}, m^{2k} - 1}]_{k=1}^{\infty} = \frac{\phi(dm, d, -1/(dmn))}{\phi(-1/n, d, -1/(dmn))}. \tag{2.39}$$

If $m > 2$ and $n > 1$ then

$$[1, \overline{d^{2k-2}n^{2k-1} - 1, 1, m^{2k-1} - 2, 1, d^{2k-1}n^{2k} - 1, m^{2k}}]_{k=1}^{\infty} = \frac{\phi(-dm, d, -1/(dmn))}{\phi(1/n, d, -1/(dmn))}. \tag{2.40}$$

If $n > 2$ and $m > 1$ then

$$[1, \overline{d^{2k-2}n^{2k-1}, m^{2k-1} - 1, 1, d^{2k-1}n^{2k} - 2, 1, m^{2k} - 1}]_{k=1}^{\infty} = \frac{\phi(-dm, -d, -1/(dmn))}{\phi(1/n, -d, -1/(dmn))}. \tag{2.41}$$

Proof. Equation 2.38 is a restatement of (2.37), with the stated conditions on m, n and d. Equations 2.39, 2.40 and 2.41 follow from 2.38 by replacing, respectively, n by $-n$, m by $-m$, d by $-d$ and clearing the negatives from the continued fraction expansions. Other combinations of sign changes will produce different regular continued fraction expansions. □

If we replace q by q^2 and c by c/q in Equation 2.35 and then apply Corollary 2, with once again c_1 arbitrary, we get that

$$\begin{aligned}
\frac{\phi(c/q, d, q^2)}{\phi(cq, d, q^2)} &= 1 + \frac{cq}{1} + \frac{dq^2}{1} + \frac{cq^3}{1} + \frac{dq^4}{1} + \frac{cq^5}{1} + \frac{dq^6}{1} + \cdots \\
&= 1 - c_1 + \frac{c_1}{1} - \frac{cq/c_1}{1} + \frac{cq/c_1}{1} - \frac{c_1 dq/c}{1} + \frac{c_1 dq/c}{1} \\
&\quad - \frac{(cq)^2/(c_1 d)}{1} + \frac{(cq)^2/(c_1 d)}{1} - \frac{c_1(dq/c)^2}{1} + \frac{c_1(dq/c)^2}{1} - \cdots \\
&\quad - \frac{(cq)^n/(c_1 d^{n-1})}{1} + \frac{(cq)^n/(c_1 d^{n-1})}{1} - \frac{c_1(dq/c)^n}{1} + \frac{c_1(dq/c)^n}{1} - \cdots \\
&= 1 - c_1 + \frac{1}{1/c_1} - \frac{1}{c_1^2/(cq)} + \frac{1}{1/c_1} - \frac{1}{c/(dq)} \\
&\quad + \frac{1}{1/c_1} - \frac{1}{c_1^2 d/(cq)^2} + \frac{1}{1/c_1} - \frac{1}{(c/(dq))^2} + \\
&\quad \cdots \\
&\quad + \frac{1}{1/c_1} - \frac{1}{c_1^2 d^{n-1}/(cq)^n} + \frac{1}{1/c_1} - \frac{1}{(c/(dq))^n} + \cdots .
\end{aligned} \tag{2.42}$$

The second equation is valid once again by Worpitzky's Theorem, which gives that the second continued fraction converges, and hence must equal its odd part.

If we now set $c_1 = 1/p$, $c = d\sqrt{m/n}$ and $q = 1/\sqrt{mn}$, we have, once again after a small manipulation to bring all the negative signs into the denominators, that

$$\frac{\phi(dm, d, 1/(mn))}{\phi(d/n, d, 1/(mn))} = -\frac{1}{p} + \left[1; \overline{p, \frac{-n^k}{p^2 d}, -p, m^k}\right]_{k=1}^{\infty}. \tag{2.43}$$

As with previous identities, we now choose various combinations of signs for the variables to get the following theorem.

Theorem 9. *Let d be a positive rational and let m, n and p be integers such that $mn > 1$ and $p^2 d | n$. If $m > 1$, $n/(p^2 d) > 1$ and $p > 1$, then*

$$\left[1; \overline{p-1, 1, \frac{n^k}{p^2 d} - 1, p-1, 1, m^k - 1}\right]_{k=1}^{\infty} = \frac{1}{p} + \frac{\phi(dm, d, 1/(mn))}{\phi(d/n, d, 1/(mn))}. \tag{2.44}$$

If $m > 1$, $n/(p^2 d) > 1$ and $p > 2$, then

$$\left[1; \overline{p, \frac{n^k}{p^2 d} - 1, 1, p-2, 1, m^k - 1}\right]_{k=1}^{\infty} = \frac{1}{p} + \frac{\phi(-dm, -d, 1/(mn))}{\phi(-d/n, -d, 1/(mn))}. \tag{2.45}$$

If $m > 1$, $n/(p^2 d) > 1$ and $p > 1$, then

$$\left[0; \overline{1, p-1, \frac{n^k}{p^2 d} - 1, 1, p-1, m^k - 1}\right]_{k=1}^{\infty} = \frac{-1}{p} + \frac{\phi(dm, d, 1/(mn))}{\phi(d/n, d, 1/(mn))}. \tag{2.46}$$

If $m > 1$, $n/(p^2 d) > 1$ and $p > 2$, then

$$\left[1; \overline{p-1, 1, \frac{n^{2k-1}}{p^2 d} - 1, p, m^{2k-1} - 1, 1, p-2, 1, \frac{n^{2k}}{p^2 d} - 1, p-1, 1, m^{2k} - 1}\right]_{k=1}^{\infty} = \frac{1}{p} + \frac{\phi(-dm, d, -1/(mn))}{\phi(d/n, d, -1/(mn))}. \tag{2.47}$$

If $m > 1$, $n/(p^2 d) > 1$ and $p > 2$, then

$$\left[1; \overline{p, \frac{n^{2k-1}}{p^2 d} - 1, 1, p-2, 1, m^{2k-1} - 1, p-1, 1, \frac{n^{2k}}{p^2 d} - 1, p-1, 1, m^{2k} - 1}\right]_{k=1}^{\infty} = \frac{1}{p} + \frac{\phi(dm, d, -1/(mn))}{\phi(-d/n, d, -1/(mn))}. \tag{2.48}$$

Proof. Equations 2.44 to 2.48 follow from Equation 2.43 by, respectively, i) keeping the sign of all variables unchanged, ii) replacing d by $-d$, iii) replacing p by $-p$, iv) replacing m by $-m$, v) replacing n by $-n$ and in each case removing the negative signs from the resulting continued fraction.

Other combinations of signs will lead to other regular continued fraction expansions. □

3. Concluding Remarks

Several of the continued fraction identities stated by Ramanujan have been generalized by various authors and it is possible that some of these can also be manipulated to give new classes of regular continued fraction expansions whose limits can represented in other ways.

In [4] the authors prove that if $0 < |q| < 1$ and is algebraic, then the Rogers-Ramanujan continued fraction $K(q)$ at (2.2) converges to a transcendental number. This means that, in Theorem 1, if we set $c = a^j$ and $d = a^k$, where j and k are non-negative integers, then the values of all of the resulting continued fractions are transcendental. Similarly, setting $r = s$ in Theorem 4 gives that the values of all of the resulting continued fractions are transcendental. It is likely that transcendence results for other q-continued fractions may be used to show the transcendence of the values of some of the other regular continued fractions in the paper, but we have not pursued that here.

References

[1] George E. Andrews,Bruce C. Berndt, Jaebum Sohn, Ae Ja Yee and Alexandru Zaharescu, *On Ramanujan's continued fraction for* $(q^2;q^3)_\infty/(q;q^3)_\infty$. Trans. Amer. Math. Soc. **355** (2003), no. 6, 2397–2411 (electronic).

[2] Bruce C. Berndt, Ramanujan's notebooks. Part III. Springer-Verlag, New York, 1991. xiv+510 pp.

[3] Bruce C. Berndt, Ramanujan's notebooks. Part V. Springer-Verlag, New York, 1998. xiv+624 pp.

[4] Daniel Duverney, Keiji Nishioka, Kumiko Nishioka and Iekata Shiokawa, *Transcendence of Rogers-Ramanujan continued fraction and reciprocal sums of Fibonacci numbers.* Proc. Japan Acad. Ser. A Math. Sci. **73** (1997), no. 7, 140–142.

[5] Takao Komatsu, *On Tasoev's continued fractions.* Math. Proc. Cambridge Philos. Soc. **134** (2003), no. 1, 1–12.

[6] Takao Komatsu, *On Hurwitzian and Tasoev's continued fractions.* Acta Arith. **107** (2003), no. 2, 161–177.

[7] Lehmer, D. H. *Continued fractions containing arithmetic progressions.* Scripta Math. **29**, 17–24. (1973).

[8] Lisa Lorentzen and Haakon Waadeland, *Continued Fractions with Applications*, North-Holland, Amsterdam-London-New York-Tokyo, 1992.

[9] Ramanujan, S. *Notebooks* (2 volumes), Tata Institute of Fundamental Research, Bombay, 1957.

[10] L.J.Rogers, *Second memoir on the expansion of certain infinite products*, Proc. London Math.Soc. **25** (1894),pp. 318-343.

[11] B. G. Tasoev, *Certain problems in the theory of continued fractions.* (Russian) Trudy Tbiliss. Univ. Mat. Mekh. Astronom. No. 16-17 (1984), 53–83.

[12] B. G. Tasoev, *On rational approximations of some numbers.* (Russian) Mat. Zametki **67** (2000), no. 6, 931–937; translation in Math. Notes **67** (2000), no. 5-6, 786–791.

[13] A. J. van der Poorten, *Explicit Formulas for Units in Certain Quadratic Number Fields.* Algorithmic number theory (Ithaca, NY, 1994), 194–208, Lecture Notes in Comput. Sci., 877, Springer, Berlin, 1994.

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