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Recommended Citation

Bowman, D., McLaughlin, J., & Wyshinski, N. (2010). Continued Fraction Proofs of m-versions of Some Identities of Rogers-Ramanujan-Slater Type. *The Ramanujan Journal*, 25(2), 203-227. Retrieved from https://digitalcommons.wcupa.edu/math_facpub/47

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CONTINUED FRACTION PROOFS OF m -VERSIONS OF SOME IDENTITIES OF ROGERS-RAMANUJAN-SLATER TYPE

DOUGLAS BOWMAN, JAMES MC LAUGHLIN, AND NANCY J. WYSHINSKI

ABSTRACT. We derive two general transformations for certain basic hypergeometric series from the recurrence formulae for the partial numerators and denominators of two q -continued fractions previously investigated by the authors.

By then specializing certain free parameters in these transformations, and employing various identities of Rogers-Ramanujan type, we derive m -versions of these identities. Some of the identities thus found are new, and some have been derived previously by other authors, using different methods.

By applying certain transformations due to Watson, Heine and Ramanujan, we derive still more examples of such m -versions of Rogers-Ramanujan-type identities.

1. INTRODUCTION

In [6], the authors prove the following generalization of the well-known Rogers-Ramanujan identities. For an integer $m \geq 0$,

$$(1.1) \quad \sum_{n=0}^{\infty} \frac{q^{n^2+mn}}{(q; q)_n} = \frac{(-1)^m q^{-m(m-1)/2} a_m(q)}{(q, q^4; q^5)} + \frac{(-1)^{m+1} q^{-m(m-1)/2} b_m(q)}{(q^2, q^3; q^5)},$$

where $a_0(q) = 1$, $b_0(q) = 0$, and for $m \geq 1$,

$$a_m(q) = \sum_n q^{n^2+n} \begin{bmatrix} m-2-n \\ n \end{bmatrix},$$

$$b_m(q) = \sum_n q^{n^2} \begin{bmatrix} m-1-n \\ n \end{bmatrix}.$$

The cases $m = 0$ and $m = 1$ give the original Rogers-Ramanujan identities, and following the authors in [12], we refer to (1.1) as an “ m -version” of the Rogers-Ramanujan identities.

We will use the term “ m -version” in the paper to mean an identity involving an integer parameter m , such that setting $m = 0$ or $m = 1$ will recover

2000 *Mathematics Subject Classification*. Primary: 33D15. Secondary: 11A55.

Key words and phrases. Continued Fractions; q -continued fraction; Rogers-Ramanujan identities; Slater’s Identities; q -Series; m -versions.

a known q -series-product identity, such as may be found on the “Slater list” (see [16] or [15]) or elsewhere.

The authors in [6] derived (1.1) by evaluating an integral involving q -Hermite polynomials in two different ways and equating the results. In [3], the authors derive (1.1) using their method of Engel Expansions, and indeed give a polynomial generalization of (1.1), similar in nature to those found at (2.9) and (3.7) below. Garrett [7] stated a more general identity that contains the polynomial identity of Andrews et. al. in [3] as a special case, and gave an m -version of another pair of identities on the Slater list [16] as another application. Other applications of Engel expansions to produce similar identities are given in [4].

In [11], determinant methods are used to derive a generalization of (1.1), as well as an m -version of Identities **38** and **39** in [16] by Slater (for a different m -version of these identities, see Corollary 14 below). In [9], determinant methods are further explored to derive m -versions of several other Slater identities, making use of the finitizations of these Slater identities given in [15] by Sills. The methods employed in [9] also allow the authors to derive m versions of certain *triples* of identities given in [16].

In [12], Ismail and Stanton gave a formula for the right side of (1.1) in the case where m is a negative integer, and, amongst other results, gave several other similar m -versions of Slater identities in the case where m is a negative integer.

In the present paper we show that m -versions of several pairs of identities of Rogers - Ramanujan type may be derived easily from known properties of two quite general q -continued fractions previously investigated by the present authors. Continued fraction methods give the easiest approach to proving this type of identity in many cases, as once the general result from the q -continued is made explicit, all that is necessary to produce m -versions of certain identities of Rogers-Ramanujan type is to specialize various parameters.

We derive still further m -version identities by applying transformations due to Watson, Heine and Ramanujan to the m -version identities we derived using continued fraction methods.

We also derive negative m -versions of several identities, by similar methods.

2. A GENERAL BASIC HYPERGEOMETRIC TRANSFORMATION

In this section we prove a general transformation (Theorem 2 below) for certain basic hypergeometric series, a transformation which gives (1.1) and several similar formulae as special cases.

We first recall some properties of continued fractions which will be used later. Let P_n denote the n -th numerator convergent, and Q_n denote the n -th denominator convergent, of the continued fraction

$$b_0 + \frac{a_1}{b_1 + \frac{a_2}{b_2 + \frac{a_3}{b_3} + \cdots}}$$

Then (see, for example, [13], p.9) the P_n 's and Q_n 's satisfy the following recurrence relations.

$$(2.1) \quad P_n = b_n P_{n-1} + a_n P_{n-2},$$

$$Q_n = b_n Q_{n-1} + a_n Q_{n-2}.$$

It is also well known (see also [13], p.9) that, for $n \geq 1$,

$$(2.2) \quad P_n Q_{n-1} - P_{n-1} Q_n = (-1)^{n-1} \prod_{i=1}^n a_i.$$

We also recall the q -binomial theorem ([1], pp. 35–36).

Lemma 1. *If $\begin{bmatrix} n \\ m \end{bmatrix}$ denotes the Gaussian polynomial defined by*

$$\begin{bmatrix} n \\ m \end{bmatrix} := \begin{bmatrix} n \\ m \end{bmatrix}_q := \begin{cases} \frac{(q; q)_n}{(q; q)_m (q; q)_{n-m}}, & \text{if } 0 \leq m \leq n, \\ 0, & \text{otherwise,} \end{cases}$$

then

$$(2.3) \quad \begin{aligned} (z; q)_N &= \sum_{j=0}^N \begin{bmatrix} N \\ j \end{bmatrix} (-1)^j z^j q^{j(j-1)/2}, \\ \frac{1}{(z; q)_N} &= \sum_{j=0}^{\infty} \begin{bmatrix} N+j-1 \\ j \end{bmatrix} z^j. \end{aligned}$$

In [5] the following result (slightly rephrased) was proven.

Theorem 1. *Let*

$$H(a, b, c, d, q) = \frac{1}{1 + \frac{-ab + cq}{a + b + dq} + \frac{-ab + cq^2}{a + b + dq^2} + \cdots + \frac{-ab + cq^n}{a + b + dq^n} + \cdots}.$$

Let $A_N := A_N(q)$ and $B_N := B_N(q)$ denote the N -th numerator convergent and N -th denominator convergent, respectively, of $H(a, b, c, d, q)$. Then A_N and B_N are given explicitly by the following formulae.

$$(2.4) \quad \begin{aligned} A_N &= b^{N-1} \sum_{n \geq 0} (d/b)^n q^{n(n+1)/2} \sum_{j \geq 0} \begin{bmatrix} n+j \\ j \end{bmatrix} (a/b)^j \\ &\quad \times \sum_{l \geq 0} \begin{bmatrix} N-1-j-l \\ n \end{bmatrix} q^{l(l-1)/2} \left(\frac{cq}{bd}\right)^l \begin{bmatrix} n \\ l \end{bmatrix}. \end{aligned}$$

For $N \geq 2$,

$$(2.5) \quad B_N = A_N + b^{N-1}(cq/b - a) \sum_{n \geq 0} (d/b)^n q^{n(n+3)/2} \sum_{j \geq 0} \begin{bmatrix} n+j \\ j \end{bmatrix} (a/b)^j \\ \times \sum_{l \geq 0} \begin{bmatrix} N-2-j-l \\ n \end{bmatrix} q^{l(l-1)/2} \left(\frac{cq}{bd}\right)^l \begin{bmatrix} n \\ l \end{bmatrix}.$$

We now prove the main result of this section.

Theorem 2. *Let x, y, z and q be complex numbers with $|q| < 1$ and $y \neq q^{-n}$, $y \neq xzq^n$ for any integer $n \geq 0$. Let $\phi(x, y, z, q)$ be defined by*

$$(2.6) \quad \phi(x, y, z, q) := \sum_{n=0}^{\infty} \frac{x^n q^{n(n+1)/2} (-z; q)_n}{(y; q)_{n+1} (q; q)_n},$$

and for each positive integer m , define $e_m(x, y, z, q)$ by

$$(2.7) \quad e_m(x, y, z, q) := \sum_{n \geq 0} x^n q^{n(n+1)/2} \sum_{j \geq 0} \begin{bmatrix} n+j \\ j \end{bmatrix} y^j \\ \times \sum_{l \geq 0} \begin{bmatrix} m-1-j-l \\ n \end{bmatrix} q^{l(l-1)/2} z^l \begin{bmatrix} n \\ l \end{bmatrix}.$$

Then, for $m \geq 2$,

$$(2.8) \quad \phi(xq^m, y, z, q) = \frac{e_m(x, y, z, q)\phi(xq, y, z, q) - e_{m-1}(xq, y, z, q)\phi(x, y, z, q)}{\prod_{j=1}^{m-1} (y - xzq^j)}.$$

Remarks: For ease in following the proof below, define

$$f_m(x, y, z, q) := \sum_{n \geq 0} x^n q^{n(n+3)/2} \sum_{j \geq 0} \begin{bmatrix} n+j \\ j \end{bmatrix} y^j \\ \times \sum_{l \geq 0} \begin{bmatrix} m-2-j-l \\ n \end{bmatrix} q^{l(l-1)/2} z^l \begin{bmatrix} n \\ l \end{bmatrix} \\ = e_{m-1}(xq, y, z, q).$$

Note also that Lemma 1 gives, for $|y| < 1$, that

$$\lim_{m \rightarrow \infty} e_m(x, y, z, q) = \phi(x, y, z, q).$$

Also for ease of notation in the proof below, we define

$$e_{n,m} := e_n(xq^m, y, z, q), \\ f_{n,m} := f_n(xq^m, y, z, q)$$

Proof of Theorem 2. From Theorem 1, the n -th numerator convergent of the continued fraction

$$\frac{-y + zx}{y + 1 + xq} + \frac{-y + zxq}{y + 1 + xq^2} + \cdots + \frac{-y + zxq^{n-1}}{y + 1 + xq^n} + \cdots$$

is $(zx-y)f_{n+1,0}$, and the n -th denominator convergent is $e_{n+1,0}$, upon noting that

$$\begin{aligned} A_{n+1} &= e_{n+1}(x, y, z, q) = e_{n+1,0}, \\ B_{n+1} - A_{n+1} &= (zx - y)f_{n+1}(x, y, z, q) = (zx - y)f_{n+1,0}. \end{aligned}$$

Thus

$$\begin{aligned} \frac{(zx - y)f_{n+m+1,0}}{e_{n+m+1,0}} &= \frac{-y + zx}{y + 1 + xq} + \cdots + \frac{-y + zxq^{m-1}}{y + 1 + xq^m} \\ &\quad + \frac{-y + zxq^m}{y + 1 + xq^{m+1}} + \cdots + \frac{-y + zxq^{m+n-1}}{y + 1 + xq^{m+n}} \\ &= \frac{-y + zx}{y + 1 + xq} + \cdots + \frac{-y + zxq^{m-1}}{y + 1 + xq^m} + \frac{(zxq^m - y)f_{n+1,m}}{e_{n+1,m}} \end{aligned}$$

Thus, by (2.1),

$$\begin{aligned} (2.9) \quad (zx - y)f_{n+m+1,0} &= [e_{n+1,m}][(zx - y)f_{m+1,0}] \\ &\quad + [(zxq^m - y)f_{n+1,m}][(zx - y)f_{m,0}], \\ e_{n+m+1,0} &= [e_{n+1,m}][e_{m+1,0}] + [(zxq^m - y)f_{n+1,m}][e_{m,0}]. \end{aligned}$$

Next, divide through the first equation by $zx - y$, let $n \rightarrow \infty$ (here taking $|y| < 1$), and use (2.9) to get

$$\begin{aligned} (2.10) \quad \phi(xq, y, z, q) &= \phi(xq^m, y, z, q)f_{m+1,0} \\ &\quad + (zxq^m - y)\phi(xq^{m+1}, y, z, q)f_{m,0}, \\ \phi(x, y, z, q) &= \phi(xq^m, y, z, q)e_{m+1,0} \\ &\quad + (zxq^m - y)\phi(xq^{m+1}, y, z, q)e_{m,0}. \end{aligned}$$

We solve this last pair of equations for $\phi(xq^{m+1}, y, z, q)$ and $\phi(xq^m, y, z, q)$ to get that

$$\phi(xq^m, y, z, q) = \frac{e_{m,0}\phi(xq, y, z, q) - f_{m,0}\phi(x, y, z, q)}{f_{m+1,0}e_{m,0} - e_{m+1,0}f_{m,0}},$$

and the result follows for $|y| < 1$ upon noting that (2.2) give

$$(zx - y)f_{m+1,0}e_{m,0} - e_{m+1,0}(zx - y)f_{m,0} = (-1)^{m-1} \prod_{i=1}^m (-y + zxq^{i-1}).$$

The full result follows from the Identity Theorem, regarding each side of (2.8) as a function of y . \square

As a first application, we give a proof of the result of Garrett, Ismail and Stanton at (1.1).

Corollary 1. For $|q| < 1$ and integral $m \geq 0$,

$$(2.11) \quad \sum_{n=0}^{\infty} \frac{q^{n^2+mn}}{(q; q)_n} = \frac{(-1)^m q^{-m(m-1)/2} a_m(q)}{(q, q^4; q^5)} + \frac{(-1)^{m+1} q^{-m(m-1)/2} b_m(q)}{(q^2, q^3; q^5)},$$

where $a_0(q) = 1$, $b_0(q) = 0$, and for $m \geq 1$,

$$a_m(q) = \sum_n q^{n^2+n} \begin{bmatrix} m-2-n \\ n \end{bmatrix},$$

$$b_m(q) = \sum_n q^{n^2} \begin{bmatrix} m-1-n \\ n \end{bmatrix}.$$

Proof. In (2.8), set $z = 1/x$ and let $y, x \rightarrow 0$. Then

$$\lim_{x \rightarrow 0} f_m(x, 0, 1/x, q) = a_m(q),$$

$$\lim_{x \rightarrow 0} e_m(x, 0, 1/x, q) = b_m(q).$$

Likewise, it is not difficult to see that

$$\lim_{x \rightarrow 0} \phi(xq^m, 0, 1/x, q) = \sum_{n=0}^{\infty} \frac{q^{n^2+mn}}{(q; q)_n}.$$

Use the Rogers-Ramanujan identities to get that

$$(2.12) \quad \lim_{x \rightarrow 0} \phi(x, 0, 1/x, q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q; q)_n} = \frac{1}{(q, q^4; q^5)_{\infty}},$$

$$\lim_{x \rightarrow 0} \phi(xq, 0, x, q) = \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q; q)_n} = \frac{1}{(q^2, q^3; q^5)_{\infty}}.$$

Lastly, with the stated values for the parameters, the denominator on the right side of (2.8) now becomes $(-1)^{m-1} q^{m(m-1)/2}$. \square

We now prove a number of similar identities, giving explicit formulae for the polynomials corresponding to the $a_m(q)$ and $b_m(q)$ in the corollary above.

Corollary 2. For $|q| < 1$ and integral $m \geq 0$,

$$(2.13) \quad \sum_{n=0}^{\infty} \frac{(-q; q)_n q^{n(n-1)/2+mn}}{(q; q)_n}$$

$$= \frac{(-1)^{m-1}}{q^{m(m-1)/2}} \left[(a_m(q) - b_m(q)) \frac{(q^4; q^4)_{\infty}}{(q; q)_{\infty}} - b_m(q) \frac{(-q; q^2)_{\infty}}{(q; q^2)_{\infty}} \right],$$

where $a_0(q) = 0$, $b_0(q) = 1$, and for $m \geq 1$,

$$a_m(q) = \sum_{n,l} q^{n(n-1)/2+l(l+1)/2} \begin{bmatrix} m-1-l \\ n \end{bmatrix} \begin{bmatrix} n \\ l \end{bmatrix},$$

$$b_m(q) = \sum_{n,l} q^{n(n+1)/2+l(l+1)/2} \begin{bmatrix} m-2-l \\ n \end{bmatrix} \begin{bmatrix} n \\ l \end{bmatrix}.$$

Proof. In (2.8), set $x = 1/q$, $y = 0$, $z = q$ and use the identities (see **A.8** and **A.13** in [15])

$$\begin{aligned}\phi(1/q, 0, q, q) &= \sum_{n=0}^{\infty} \frac{(-q; q)_n q^{n(n-1)/2}}{(q; q)_n} = \frac{(q^4; q^4)_{\infty}}{(q; q)_{\infty}} + \frac{(-q; q^2)_{\infty}}{(q; q^2)_{\infty}}, \\ \phi(1, 0, q, q) &= \sum_{n=0}^{\infty} \frac{(-q; q)_n q^{n(n+1)/2}}{(q; q)_n} = \frac{(q^4; q^4)_{\infty}}{(q; q)_{\infty}}.\end{aligned}$$

□

Corollary 3. For $|q| < 1$ and integral $m \geq 0$,

$$(2.14) \quad \sum_{n=0}^{\infty} \frac{q^{n^2+2mn}}{(q^4; q^4)_n} = (-1)^{m-1} \left[\frac{a_m(q)}{(q^2, q^3; q^5)_{\infty} (-q^2; q^2)_{\infty}} - \frac{b_m(q)}{(q, q^4; q^5)_{\infty} (-q^2; q^2)_{\infty}} \right],$$

where $a_0(q) = 0$, $b_0(q) = 1$, and for $m \geq 1$,

$$\begin{aligned}a_m(q) &= \sum_{n,j} q^{n^2} (-1)^j \begin{bmatrix} m-1-j \\ n \end{bmatrix}_{q^2} \begin{bmatrix} n+j \\ j \end{bmatrix}_{q^2}, \\ b_m(q) &= \sum_{n,j} q^{n^2+2n} (-1)^j \begin{bmatrix} m-2-j \\ n \end{bmatrix}_{q^2} \begin{bmatrix} n+j \\ j \end{bmatrix}_{q^2}.\end{aligned}$$

Proof. In (2.8), replace q with q^2 , set $x = 1/q$, $y = -1$, $z = 0$ and use the identities (see **A.16** and **A.20** in [15])

$$(2.15) \quad \begin{aligned}\phi(1/q, -1, 0, q^2) &= \sum_{n=0}^{\infty} \frac{q^{n^2}}{2(q^4; q^4)_n} = \frac{1}{2(q, q^4; q^5)_{\infty} (-q^2; q^2)_{\infty}}, \\ \phi(q, -1, 0, q^2) &= \sum_{n=0}^{\infty} \frac{q^{n^2+2n}}{2(q^4; q^4)_n} = \frac{1}{2(q, q^4; q^5)_{\infty} (-q^2; q^2)_{\infty}}.\end{aligned}$$

The result follows after cancelling the “2” factor in the denominators. □

The next corollary involves the analytic versions of the Göllnitz-Gordon identities.

Corollary 4. For $|q| < 1$ and integral $m \geq 0$,

$$(2.16) \quad \sum_{n=0}^{\infty} \frac{(-q; q^2)_n q^{n^2+2mn}}{(q^2; q^2)_n} = \frac{(-1)^{m-1}}{q^{m(m-1)}} \left[\frac{a_m(q)}{(q^3, q^4, q^5; q^8)_{\infty}} - \frac{b_m(q)}{(q, q^4, q^7; q^8)_{\infty}} \right],$$

where $a_0(q) = 0$, $b_0(q) = 1$, and for $m \geq 1$,

$$a_m(q) = \sum_{n,l} q^{n^2+l^2} \begin{bmatrix} m-1-l \\ n \end{bmatrix}_{q^2} \begin{bmatrix} n \\ l \end{bmatrix}_{q^2},$$

$$b_m(q) = \sum_{n,l} q^{n^2+2n+l^2} \begin{bmatrix} m-2-l \\ n \end{bmatrix}_{q^2} \begin{bmatrix} n \\ l \end{bmatrix}_{q^2}.$$

Proof. In (2.8), replace q with q^2 , set $x = 1/q$, $y = 0$, $z = q$ and use the identities (see **A.34** and **A.36** in [15])

$$(2.17) \quad \phi(1/q, 0, q, q^2) = \sum_{n=0}^{\infty} \frac{(-q; q^2)_n q^{n^2}}{(q^2; q^2)_n} = \frac{1}{(q, q^4, q^7; q^8)_{\infty}},$$

$$\phi(q, 0, q, q^2) = \sum_{n=0}^{\infty} \frac{(-q; q^2)_n q^{n^2+2n}}{(q^2; q^2)_n} = \frac{1}{(q^3, q^4, q^5; q^8)_{\infty}}.$$

□

2.1. Implications of Watson's Transformation. We next recall Watson's transformation:

$${}_8\phi_7 \left(\begin{matrix} A, q\sqrt{A}, -q\sqrt{A}, B, C, D, E, q^{-n} \\ \sqrt{A}, -\sqrt{A}, Aq/B, Aq/C, Aq/D, Aq/E, Aq^{n+1}; q, \frac{A^2 q^{n+2}}{BCDE} \end{matrix} \right) =$$

$$\frac{(Aq)_n (Aq/DE)_n}{(Aq/D)_n (Aq/E)_n} {}_4\phi_3 \left(\begin{matrix} Aq/BC, D, E, q^{-n} \\ Aq/B, Aq/C, DEq^{-n}/A; q, q \end{matrix} \right),$$

where n is a non-negative integer. If we let B, D and $n \rightarrow \infty$ (as in [10]), replace A with zx , C with zx/y and E with $-z$, and multiply both sides by $1/(1-y)$, we get

$$(2.18) \quad \sum_{n \geq 0} \frac{(1 - zxq^{2n})(zx, zx/y, -z; q)_n (xy)^n q^{n(3n+1)/2}}{(1 - zx)(-xq, q; q)_n (y; q)_{n+1}}$$

$$= \frac{(zxq; q)_{\infty}}{(-xq; q)_{\infty}} \sum_{n \geq 0} \frac{x^n q^{n(n+1)/2} (-z; q)_n}{(y; q)_{n+1} (q; q)_n}.$$

Notice that the series on the right is the series $\phi(x, y, z, q)$ from (2.6), so that the special case of Watson's transformation at (2.18) may be used in conjunction with the specializations of x, y and z in Corollaries 1 - 4 to produce a new set of summation formulae.

Corollary 5. For $|q| < 1$ and integral $m \geq 0$,

$$(2.19) \quad \sum_{n=0}^{\infty} \frac{(1 - q^{2n+m})(q; q)_{n+m-1} q^{n(5n-1)/2+2mn} (-1)^n}{(q; q)_n}$$

$$= \frac{b_m(q)(q, q^4, q^5; q^5)_{\infty} - a_m(q)(q^2, q^3, q^5; q^5)_{\infty}}{(-1)^{m-1} q^{m(m-1)/2}},$$

where $a_0(q) = 1$, $b_0(q) = 0$, and for $m \geq 1$,

$$a_m(q) = \sum_n q^{n^2+n} \begin{bmatrix} m-2-n \\ n \end{bmatrix},$$

$$b_m(q) = \sum_n q^{n^2} \begin{bmatrix} m-1-n \\ n \end{bmatrix}.$$

Proof. In (2.18), replace x with xq^m , set $z = 1/x$ and then let $y, x \rightarrow 0$. Combine the resulting identity with (2.11), and (2.19) follows. \square

Corollary 6. For $|q| < 1$ and integral $m \geq 0$,

$$(2.20) \quad \sum_{n=0}^{\infty} \frac{(1-q^{2n+m})(-q; q)_n (q; q)_{n+m-1} q^{2n^2-n+2mn} (-1)^n}{(q; q)_n (-q; q)_{n+m-1}}$$

$$= \frac{(-1)^{m-1}}{q^{m(m-1)/2}} \left[(a_m(q) - b_m(q)) \frac{(q^4; q^4)_{\infty}}{(-q; q)_{\infty}} - b_m(q) \frac{(q^2; q^2)_{\infty}}{(-q^2; q^2)_{\infty}} \right],$$

where $a_0(q) = 0$, $b_0(q) = 1$, and for $m \geq 1$,

$$a_m(q) = \sum_{n,l} q^{n(n-1)/2+l(l+1)/2} \begin{bmatrix} m-1-l \\ n \end{bmatrix} \begin{bmatrix} n \\ l \end{bmatrix},$$

$$b_m(q) = \sum_{n,l} q^{n(n+1)/2+l(l+1)/2} \begin{bmatrix} m-2-l \\ n \end{bmatrix} \begin{bmatrix} n \\ l \end{bmatrix}.$$

Proof. This time in (2.18), replace x with xq^m , and then set $x = 1/q$, $z = q$ and let $y \rightarrow 0$. Combine the resulting identity with (2.13), and (2.20) follows. \square

Corollary 7. For $|q| < 1$ and integral $m \geq 0$,

$$(2.21) \quad \sum_{n=0}^{\infty} \frac{q^{3n^2+2mn} (-1)^n}{(-q; q^2)_{m+n} (q^4; q^4)_n}$$

$$= (-1)^{m-1} \left[\frac{a_m(q)}{(q^2, q^3; q^5)_{\infty} (-q; q)_{\infty}} - \frac{b_m(q)}{(q, q^4; q^5)_{\infty} (-q; q)_{\infty}} \right],$$

where $a_0(q) = 0$, $b_0(q) = 1$, and for $m \geq 1$,

$$a_m(q) = \sum_{n,j} q^{n^2} (-1)^j \begin{bmatrix} m-1-j \\ n \end{bmatrix}_{q^2} \begin{bmatrix} n+j \\ j \end{bmatrix}_{q^2},$$

$$b_m(q) = \sum_{n,j} q^{n^2+2n} (-1)^j \begin{bmatrix} m-2-j \\ n \end{bmatrix}_{q^2} \begin{bmatrix} n+j \\ j \end{bmatrix}_{q^2}.$$

Proof. In (2.18), replace q with q^2 and x with xq^{2m} , and then set $x = 1/q$, $y = -1$ and $z = 0$. Combine the resulting identity with (2.14), and now (2.21) follows. \square

Corollary 8. For $|q| < 1$ and integral $m \geq 0$,

$$(2.22) \quad \sum_{n=0}^{\infty} \frac{(1 - q^{4n+2m})(-q; q^2)_n (q^2; q^2)_{n+m-1} (-1)^n q^{4n^2 - n + 4mn}}{(q^2; q^2)_n (-q; q^2)_{n+m}} \\ = \frac{(-1)^{m-1}}{q^{m(m-1)}} \left[a_m(q)(q, q^7, q^8; q^8)_{\infty} - b_m(q)(q^3, q^5, q^8; q^8)_{\infty} \right],$$

where $a_0(q) = 0$, $b_0(q) = 1$, and for $m \geq 1$,

$$a_m(q) = \sum_{n,l} q^{n^2+l^2} \begin{bmatrix} m-1-l \\ n \end{bmatrix}_{q^2} \begin{bmatrix} n \\ l \end{bmatrix}_{q^2}, \\ b_m(q) = \sum_{n,l} q^{n^2+2n+l^2} \begin{bmatrix} m-2-l \\ n \end{bmatrix}_{q^2} \begin{bmatrix} n \\ l \end{bmatrix}_{q^2}.$$

Proof. This time in (2.18), replace q with q^2 and x with xq^{2m} , and then set $x = 1/q$, $y = 0$ and $z = q$. Combine the resulting identity with (2.16), and (2.22) follows after some simple q -product manipulations. \square

2.2. Implications of Heine's Transformation. A number of other transformations may be employed to derive new summation formulae, in ways that are similar to how Watson's transformation was used above.

The first of these is Heine's transformation:

$$\sum_{n=0}^{\infty} \frac{(a, b; q)_n t^n}{(c, q; q)_n} = \frac{(b, at; q)_{\infty}}{(c, t; q)_{\infty}} \sum_{n=0}^{\infty} \frac{(c/b, t; q)_n b^n}{(at, q; q)_n}.$$

If we replace a with $-xq/t$, b with $-z$ and c with yq , multiple both sides by $1/(1-y)$ and let $t \rightarrow 0$, then the following transformation results.

$$(2.23) \quad \sum_{n \geq 0} \frac{x^n q^{n(n+1)/2} (-z; q)_n}{(y; q)_{n+1} (q; q)_n} = \frac{(-z, -xq; q)_{\infty}}{(y; q)_{\infty}} \sum_{n \geq 0} \frac{(-qy/z; q)_n (-z)^n}{(-xq; q)_n (q; q)_n}.$$

Note that the series on the left is the series $\phi(x, y, z, q)$ from (2.6), so this transformation may be used in conjunction with Corollaries 2 - 4 to produce new summation formulae.

Corollary 9. For $|q| < 1$ and integral $m \geq 0$,

$$(2.24) \quad \sum_{n=0}^{\infty} \frac{(-q)^n}{(-q; q)_{m+n-1} (q; q)_n} \\ = \frac{(-1)^{m-1}}{q^{m(m-1)/2}} \left[\frac{(a_m(q) - b_m(q))}{(-q; q^2)_{\infty}} - \frac{b_m(q)}{(-q^2; q^2)_{\infty}} \right],$$

where $a_0(q) = 0$, $b_0(q) = 1$, and for $m \geq 1$,

$$a_m(q) = \sum_{n,l} q^{n(n-1)/2+l(l+1)/2} \begin{bmatrix} m-1-l \\ n \end{bmatrix} \begin{bmatrix} n \\ l \end{bmatrix},$$

$$b_m(q) = \sum_{n,l} q^{n(n+1)/2+l(l+1)/2} \begin{bmatrix} m-2-l \\ n \end{bmatrix} \begin{bmatrix} n \\ l \end{bmatrix}.$$

Proof. In (2.23), set $x = q^{m-1}$ and $y = 0$, $z = q$, and combine with (2.13). \square

Corollary 10. For $|q| < 1$ and integral $m \geq 0$,

$$(2.25) \quad \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(-q; q^2)_{m+n} (q^2; q^2)_n}$$

$$= (-1)^{m-1} \left[\frac{a_m(q)}{(q^2, q^3; q^5)_{\infty} (-q; q^2)_{\infty}} - \frac{b_m(q)}{(q, q^4; q^5)_{\infty} (-q; q^2)_{\infty}} \right],$$

where $a_0(q) = 0$, $b_0(q) = 1$, and for $m \geq 1$,

$$a_m(q) = \sum_{n,j} q^{n^2} (-1)^j \begin{bmatrix} m-1-j \\ n \end{bmatrix}_{q^2} \begin{bmatrix} n+j \\ j \end{bmatrix}_{q^2},$$

$$b_m(q) = \sum_{n,j} q^{n^2+2n} (-1)^j \begin{bmatrix} m-2-j \\ n \end{bmatrix}_{q^2} \begin{bmatrix} n+j \\ j \end{bmatrix}_{q^2}.$$

Proof. In (2.23), replace q with q^2 , set $x = q^{2m-1}$ and $y = -1$, let $z \rightarrow 0$ and combine with (2.14). \square

Corollary 11. For $|q| < 1$ and integral $m \geq 0$,

$$(2.26) \quad \sum_{n=0}^{\infty} \frac{(-q)^n}{(-q; q^2)_{m+n} (q^2; q^2)_n}$$

$$= \frac{(-1)^{m-1}}{q^{m(m-1)} (-q, -q; q^2)_{\infty}} \left[\frac{a_m(q)}{(q^3, q^4, q^5; q^8)_{\infty}} - \frac{b_m(q)}{(q, q^4, q^7; q^8)_{\infty}} \right],$$

where $a_0(q) = 0$, $b_0(q) = 1$, and for $m \geq 1$,

$$a_m(q) = \sum_{n,l} q^{n^2+l^2} \begin{bmatrix} m-1-l \\ n \end{bmatrix}_{q^2} \begin{bmatrix} n \\ l \end{bmatrix}_{q^2},$$

$$b_m(q) = \sum_{n,l} q^{n^2+2n+l^2} \begin{bmatrix} m-2-l \\ n \end{bmatrix}_{q^2} \begin{bmatrix} n \\ l \end{bmatrix}_{q^2}.$$

Proof. In (2.23), replace q with q^2 , set $x = q^{2m-1}$ and $y = 0$, $z = q$, and combine with (2.16). \square

2.3. Implications of a Transformation of Ramanujan. Another transformation we consider is one stated by Ramanujan (**Entry 2.2.3** in Chapter 2 of [2]), which also follows as a consequence of a transformation of Sears [14]:

$$\sum_{n \geq 0} \frac{b^n q^{n(n+1)/2} (-aq/b; q)_n}{(-cq; q)_n (q; q)_n} = (-bq; q)_\infty \sum_{n \geq 0} \frac{a^n q^{n(n+1)} (bc/a; q)_n}{(-bq; q)_n (-cq; q)_n (q; q)_n}.$$

If we replace b with x , c with $-y$ and a with zx/q , and multiply both sides by $1/(1-y)$, then we get

$$(2.27) \quad \sum_{n \geq 0} \frac{x^n q^{n(n+1)/2} (-z; q)_n}{(y; q)_{n+1} (q; q)_n} = (-xq; q)_\infty \sum_{n \geq 0} \frac{(zx)^n q^{n^2} (-qy/z; q)_n}{(-xq; q)_n (y; q)_{n+1} (q; q)_n}.$$

Once again, the series on the left is the series $\phi(x, y, z, q)$ from (2.6), and specializing x , y and z as in Corollaries 1 - 4 will give summation formulae similar to those above (although not all are new).

As an example of an application of this transformation, if we set $x = q^{m-1}$, $y = 0$ and $z = q$, and combine with the identity at (2.13), then the following summation formula results.

Corollary 12. For $|q| < 1$ and integral $m \geq 0$,

$$(2.28) \quad \sum_{n=0}^{\infty} \frac{q^{n^2+mn}}{(-q; q)_{m+n-1} (q; q)_n} = \frac{(-1)^{m-1}}{q^{m(m-1)/2}} \left[(a_m(q) - b_m(q)) (-q^2; q^2)_\infty - b_m(q) (-q; q^2)_\infty \right],$$

where $a_0(q) = 0$, $b_0(q) = 1$, and for $m \geq 1$,

$$a_m(q) = \sum_{n,l} q^{n(n-1)/2+l(l+1)/2} \begin{bmatrix} m-1-l \\ n \end{bmatrix} \begin{bmatrix} n \\ l \end{bmatrix},$$

$$b_m(q) = \sum_{n,l} q^{n(n+1)/2+l(l+1)/2} \begin{bmatrix} m-2-l \\ n \end{bmatrix} \begin{bmatrix} n \\ l \end{bmatrix}.$$

3. A SECOND GENERAL TRANSFORMATION

The following result was also proven in [5]:

Theorem 3. Let a, b, c, d be complex numbers with $d \neq 0$ and $|q| < 1$. Define

$$H_1(a, b, c, d, q) := \frac{1}{1 + \frac{-abq + c}{(a+b)q + d} + \cdots + \frac{-abq^{2n+1} + cq^n}{(a+b)q^{n+1} + d} + \cdots}.$$

Let $C_N := C_N(q)$ and $D_N := D_N(q)$ denote the N -th numerator convergent and N -th denominator convergent, respectively, of $H_1(a, b, c, d, q)$. Then C_N

and D_N are given explicitly by the following formulae.

$$(3.1) \quad C_N = d^{N-1} \sum_{j,l,n \geq 0} a^j b^{n-j-l} c^l d^{-n-l} q^{n(n+1)/2+l(l-1)/2} \\ \times \begin{bmatrix} N-1-n+j \\ j \end{bmatrix}_q \begin{bmatrix} N-1-j-l \\ n-j-l \end{bmatrix}_q \begin{bmatrix} N-1-n \\ l \end{bmatrix}_q.$$

For $N \geq 2$,

$$(3.2) \quad D_N = C_N + (c/bq - a) \sum_{j,l,n \geq 0} a^j b^{n+1-j-l} c^l d^{N-2-n-l} \times \\ q^{(n+1)(n+2)/2+l(l-1)/2} \begin{bmatrix} N-2-n+j \\ j \end{bmatrix}_q \begin{bmatrix} N-2-j-l \\ n-j-l \end{bmatrix}_q \begin{bmatrix} N-2-n \\ l \end{bmatrix}_q.$$

$$\lim_{N \rightarrow \infty} \frac{C_N}{d^{N-1}} = (-aq/d)_\infty \sum_{j=0}^{\infty} \frac{(b/d)^j (-c/bd)_j q^{j(j+1)/2}}{(q)_j (-aq/d)_j}, \\ \lim_{N \rightarrow \infty} \frac{D_N - C_N}{d^{N-1}} = \frac{c - abq}{d} (-aq^2/d)_\infty \sum_{j=0}^{\infty} \frac{(b/d)^j (-c/bd)_j q^{(j^2+3j)/2}}{(q)_j (-aq^2/d)_j}.$$

From the result above we derive the following general identity.

Theorem 4. Let x, y, z and q be complex numbers with $|q| < 1$ and $y \neq -q^{-n}/x$, $y \neq zq^{-n}/x$ for $n \geq 1$. Let $\Phi(x, y, z, q)$ be defined by

$$(3.3) \quad \Phi(x, y, z, q) := \sum_{n=0}^{\infty} \frac{x^n q^{n(n+1)/2} (-z; q)_n}{(-xyq; q)_n (q; q)_n},$$

and for a positive integer m , define $g_m(x, y, z, q)$ by

$$(3.4) \quad g_m(x, y, z, q) := \sum_{n,j,l \geq 0} x^n y^j z^l q^{n(n+1)/2+l(l-1)/2} \begin{bmatrix} m-1-n+j \\ j \end{bmatrix} \\ \times \begin{bmatrix} m-1-j-l \\ n-j-l \end{bmatrix} \begin{bmatrix} m-1-n \\ l \end{bmatrix}.$$

Then, for $m \geq 2$,

$$(3.5) \quad \sum_{n=0}^{\infty} \frac{x^n q^{n(n+1)/2+mn} (-z; q)_n}{(-xyq; q)_{m+n} (q; q)_n} = \frac{\Phi(xq^m, y, z, q)}{(-xyq; q)_m} = \\ \frac{g_m(x, y, z, q) \Phi(xq, y, z, q) / (1 + xyq) - g_{m-1}(xq, y, z, q) \Phi(x, y, z, q)}{\prod_{j=1}^{m-1} (x^2 y q^{2j+1} - x z q^j)}.$$

Proof. As in the proof of Theorem 2, for ease of notation we define a second polynomial sequence,

$$\begin{aligned} h_m(x, y, z, q) &:= \sum_{n, j, l \geq 0} x^n y^j z^l q^{n(n+3)/2 + l(l-1)/2} \begin{bmatrix} m-2-n+j \\ j \end{bmatrix} \\ &\quad \times \begin{bmatrix} m-2-j-l \\ n-j-l \end{bmatrix} \begin{bmatrix} m-2-n \\ l \end{bmatrix} \\ &= g_{m-1}(xq, y, z, q), \end{aligned}$$

for integral $m \geq 1$. Note that Lemma 1 (and Theorem 3 - see [5] for details) gives that

$$(3.6) \quad \begin{aligned} \lim_{m \rightarrow \infty} g_m(x, y, z, q) &= (-xyq; q)_\infty \Phi(x, y, z, q), \\ \lim_{m \rightarrow \infty} h_m(x, y, z, q) &= (-xyq^2; q)_\infty \Phi(xq, y, z, q). \end{aligned}$$

For ease of notation, we once again define

$$\begin{aligned} g_{n,m} &:= g_n(xq^m, y, z, q), \\ h_{n,m} &:= h_n(xq^m, y, z, q) \end{aligned}$$

From Theorem 3, the n -th numerator convergent of the continued fraction

$$\frac{-x^2 yq + zx}{(x + xy)q + 1} + \frac{-x^2 yq^3 + zxq}{(x + xy)q^2 + 1} + \cdots + \frac{-x^2 yq^{2n-1} + zxq^{n-1}}{(x + xy)q^n + 1} + \cdots$$

is $(zx - x^2 yq)h_{n+1,0}$, and the n -th denominator convergent is $g_{n+1,0}$.

Once again appealing to the recurrence relations for a continued fraction (see (2.1), and the proof of Theorem 2), we have that

$$(3.7) \quad \begin{aligned} (zx - x^2 yq)h_{n+m+1,0} &= [g_{n+1,m}][(zx - x^2 yq)h_{m+1,0}] \\ &\quad + [(zxq^m - x^2 yq^{2m+1})h_{n+1,m}][(zx - x^2 yq)h_{m,0}], \\ g_{n+m+1,0} &= [g_{n+1,m}][g_{m+1,0}] + [(zxq^m - x^2 yq^{2m+1})h_{n+1,m}][g_{m,0}]. \end{aligned}$$

Divide through the first equation by $zx - x^2 yq$, let $n \rightarrow \infty$ (here taking $|y| < 1$), and use (3.6) to get (after dividing through in each case by $(-xyq^{m+1}; q)_\infty$) that

$$(3.8) \quad \begin{aligned} (-xyq^2; q)_{m-1} \Phi(xq, y, z, q) &= \Phi(xq^m, y, z, q)h_{m+1,0} \\ &\quad + \frac{zxq^m - x^2 yq^{2m+1}}{1 + xyq^{m+1}} \Phi(xq^{m+1}, y, z, q)h_{m,0}, \\ (-xyq; q)_m \Phi(x, y, z, q) &= \Phi(xq^m, y, z, q)g_{m+1,0} \\ &\quad + \frac{zxq^m - x^2 yq^{2m+1}}{1 + xyq^{m+1}} \Phi(xq^{m+1}, y, z, q)g_{m,0}. \end{aligned}$$

Solve this latter pair of equations for $\Phi(xq^m, y, z, q)$ and $\Phi(xq^{m+1}, y, z, q)$ and, as in the proof of Theorem 2, the result once again follows. \square

As with Theorem 2, Theorem 4 also leads to summation formulae that are similar to that of Garrett, Ismail and Stanton at (1.1).

Corollary 13. For $|q| < 1$ and integral $m \geq 0$,

$$(3.9) \quad \sum_{n=0}^{\infty} \frac{q^{n^2+2mn}}{(q; q^2)_{m+n}(q^2; q^2)_n} = \frac{(-1)^{m-1}(-q; q^2)_{\infty}}{q^{2m(m-1)}(q^2; q^2)_{\infty}} \left[a_m(q)(q^4, q^{16}, q^{20}; q^{20})_{\infty} - b_m(q)(q^8, q^{12}, q^{20}; q^{20})_{\infty} \right],$$

where $a_0(q) = 0$, $b_0(q) = 1$, and for $m \geq 1$,

$$a_m(q) = \sum_{n,j} q^{n^2} (-1)^j \begin{bmatrix} m-1-n+j \\ j \end{bmatrix}_{q^2} \begin{bmatrix} m-1-j \\ n-j \end{bmatrix}_{q^2},$$

$$b_m(q) = \sum_{n,j} q^{n^2+2n} (-1)^j \begin{bmatrix} m-2-n+j \\ j \end{bmatrix}_{q^2} \begin{bmatrix} m-2-j \\ n-j \end{bmatrix}_{q^2}.$$

Proof. In (3.5), replace q with q^2 , set $x = 1/q$, $y = -1$, $z = 0$ and use the identities (see **A.79** and **A.96** in [15])

$$(3.10) \quad \Phi(1/q, -1, 0, q^2) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q, q^2; q^2)_n} = (q^8, q^{12}, q^{20}; q^{20})_{\infty} \frac{(-q; q^2)_{\infty}}{(q^2; q^2)_{\infty}},$$

$$\Phi(q, -1, 0, q^2) = \sum_{n=0}^{\infty} \frac{q^{n^2+2n}}{(q^3, q^2; q^2)_n} = (1-q)(q^4, q^{16}, q^{20}; q^{20})_{\infty} \frac{(-q; q^2)_{\infty}}{(q^2; q^2)_{\infty}}.$$

The result follows after some simple algebraic manipulations. □

Corollary 14. For $|q| < 1$ and integral $m \geq 0$,

$$(3.11) \quad \sum_{n=0}^{\infty} \frac{q^{2n^2+2mn}}{(q; q^2)_{m+n}(q^2; q^2)_n} = \frac{(-1)^{m-1}}{q^{m(m-1)}(q^2; q^2)_{\infty}} \left[a_m(q)(-q, -q^7, q^8; q^8)_{\infty} - b_m(q)(-q^3, -q^5, q^8; q^8)_{\infty} \right],$$

where $a_0(q) = 0$, $b_0(q) = 1$, and for $m \geq 1$,

$$a_m(q) = \sum_{n,l} q^{n^2+l^2} (-1)^{n-l} \begin{bmatrix} m-1-l \\ n-l \end{bmatrix}_{q^2} \begin{bmatrix} m-1-n \\ l \end{bmatrix}_{q^2},$$

$$b_m(q) = \sum_{n,l} q^{n^2+2n+l^2} (-1)^{n-l} \begin{bmatrix} m-2-l \\ n-l \end{bmatrix}_{q^2} \begin{bmatrix} m-2-n \\ l \end{bmatrix}_{q^2}.$$

Proof. In (3.5), replace q with q^2 , set $y = -1/(xq)$, $z = 1/x$, let $x \rightarrow 0$ and use the identities (see **A.38** and **A.39** in [15])

$$\begin{aligned}\Phi(x, -1/xq, 1/x, q^2) &\rightarrow \sum_{n=0}^{\infty} \frac{q^{2n^2}}{(q, q^2; q^2)_n} = \frac{(-q^3, -q^5, q^8; q^8)_{\infty}}{(q^2; q^2)_{\infty}}, \\ \Phi(xq^2, -1/xq, 1/x, q^2) &\rightarrow \sum_{n=0}^{\infty} \frac{q^{2n^2+2n}}{(q^3, q^2; q^2)_n} = \frac{(1-q)(-q, -q^7, q^8; q^8)_{\infty}}{(q^2; q^2)_{\infty}}.\end{aligned}$$

□

Corollary 15. For $|q| < 1$ and integral $m \geq 0$,

$$(3.12) \quad \sum_{n=0}^{\infty} \frac{(-q; q^2)_n q^{n^2+2mn}}{(q; q^2)_{m+n} (q^2; q^2)_n} = \frac{(-1)^{m-1}}{q^{m(m-1)} (-q^2; q^2)_{m-1} (q; q)_{\infty}} \\ \times \left[a_m(q) (q^2, q^{10}, q^{12}; q^{12})_{\infty} - b_m(q) (q^6, q^6, q^{12}; q^{12})_{\infty} \right],$$

where $a_0(q) = 0$, $b_0(q) = 1$, and for $m \geq 1$,

$$\begin{aligned}a_m(q) &= \sum_{n,j,l \geq 0} (-1)^j q^{n^2+l^2} \begin{bmatrix} m-1-n+j \\ j \end{bmatrix}_{q^2} \\ &\quad \times \begin{bmatrix} m-1-j-l \\ n-j-l \end{bmatrix}_{q^2} \begin{bmatrix} m-1-n \\ l \end{bmatrix}_{q^2}, \\ b_m(q) &= \sum_{n,j,l \geq 0} (-1)^j q^{n^2+2n+l^2} \begin{bmatrix} m-2-n+j \\ j \end{bmatrix}_{q^2} \\ &\quad \times \begin{bmatrix} m-2-j-l \\ n-j-l \end{bmatrix}_{q^2} \begin{bmatrix} m-2-n \\ l \end{bmatrix}_{q^2}.\end{aligned}$$

Proof. In (3.5), replace q with q^2 , set $x = 1/q$, $y = -1$, $z = q$, and use the identities (see **A.29** and **A.50** in [15])

$$\begin{aligned}\Phi(1/q, -1, q, q^2) &= \sum_{n=0}^{\infty} \frac{(-q; q^2)_n q^{n^2}}{(q, q^2; q^2)_n} = \frac{(q^6, q^6, q^{12}; q^{12})_{\infty}}{(q; q)_{\infty}}, \\ \Phi(q, -1, q, q^2) &= \sum_{n=0}^{\infty} \frac{(-q; q^2)_n q^{n^2+2n}}{(q^3, q^2; q^2)_n} = \frac{(1-q)(q^2, q^{10}, q^{12}; q^{12})_{\infty}}{(q; q)_{\infty}}.\end{aligned}$$

□

The transformations of Watson, Heine and Ramanujan that were used in conjunction with the series in Theorem 2 in the previous section to produce new summation formulae may similarly be used in conjunction with the series in Theorem 4 (although not all the resulting summations are new). We first consider Watson's transformation.

3.1. Watson's Transformation Again. If we cancel a factor of $1/(1-y)$ in (2.18) and then replace y with $-xy$, we get the transformation

$$(3.13) \quad \sum_{n \geq 0} \frac{(1-zxq^{2n})(zx, -z/y, -z; q)_n (-x^2y)^n q^{n(3n+1)/2}}{(1-zx)(-xq, q; q)_n (-xyq; q)_n} \\ = \frac{(zxq; q)_\infty}{(-xq; q)_\infty} \sum_{n \geq 0} \frac{x^n q^{n(n+1)/2} (-z; q)_n}{(-xyq; q)_n (q; q)_n},$$

where the series on the right is the series $\Phi(x, y, z, q)$ from Theorem 4.

Corollary 16. For $|q| < 1$ and integral $m \geq 0$,

$$(3.14) \quad \sum_{n=0}^{\infty} \frac{q^{3n^2-n+4mn}}{(q^2; q^4)_{m+n} (q^2; q^2)_n} = \\ \frac{(-1)^{m-1}}{q^{2m(m-1)} (q^2; q^2)_\infty} \left[a_m(q)(q^4, q^{16}, q^{20}; q^{20})_\infty - b_m(q)(q^8, q^{12}, q^{20}; q^{20})_\infty \right],$$

where $a_0(q) = 0$, $b_0(q) = 1$, and for $m \geq 1$,

$$a_m(q) = \sum_{n,j} q^{n^2} (-1)^j \begin{bmatrix} m-1-n+j \\ j \end{bmatrix}_{q^2} \begin{bmatrix} m-1-j \\ n-j \end{bmatrix}_{q^2}, \\ b_m(q) = \sum_{n,j} q^{n^2+n} (-1)^j \begin{bmatrix} m-2-n+j \\ j \end{bmatrix}_{q^2} \begin{bmatrix} m-2-j \\ n-j \end{bmatrix}_{q^2}.$$

Proof. Replace q with q^2 in (3.13), set $x = q^{2m-1}$, $y = -1$, $z = 0$, combine the resulting identity with (3.9), and the result follows after some simple q -product manipulations. \square

Corollary 17. For $|q| < 1$ and integral $m \geq 0$,

$$(3.15) \quad \sum_{n=0}^{\infty} \frac{(1-q^{4n+2m})(q^2; q^4)_n (q^2; q^2)_{m+n-1} q^{3n^2-n+4mn}}{(q^2; q^4)_{m+n} (q^2; q^2)_n} = \\ \frac{(-1)^{m-1} (-q^2; q^2)_\infty}{q^{m(m-1)} (-q^2; q^2)_{m-1}} \left[a_m(q)(q^2, q^{10}, q^{12}; q^{12})_\infty - b_m(q)(q^6, q^6, q^{12}; q^{12})_\infty \right],$$

where $a_0(q) = 0$, $b_0(q) = 1$, and for $m \geq 1$,

$$a_m(q) = \sum_{n,j,l \geq 0} (-1)^j q^{n^2+l^2} \begin{bmatrix} m-1-n+j \\ j \end{bmatrix}_{q^2} \\ \times \begin{bmatrix} m-1-j-l \\ n-j-l \end{bmatrix}_{q^2} \begin{bmatrix} m-1-n \\ l \end{bmatrix}_{q^2}, \\ b_m(q) = \sum_{n,j,l \geq 0} (-1)^j q^{n^2+2n+l^2} \begin{bmatrix} m-2-n+j \\ j \end{bmatrix}_{q^2}$$

$$\times \begin{bmatrix} m-2-j-l \\ n-j-l \end{bmatrix}_{q^2} \begin{bmatrix} m-2-n \\ l \end{bmatrix}_{q^2}.$$

Proof. Once again replace q with q^2 in (3.13), set $x = q^{2m-1}$, $y = -1$, $z = q$, and combine the resulting identity with (3.12). \square

3.2. Heine's Transformation Again. . If we cancel a factor of $1/(1-y)$ in (2.23), replace y with $-xy$ and rearrange, we get the identity

$$(3.16) \quad \sum_{n \geq 0} \frac{(qxy/z; q)_n (-z)^n}{(-xq; q)_n (q; q)_n} = \frac{(-xyq; q)_\infty}{(-z, -xq; q)_\infty} \sum_{n \geq 0} \frac{x^n q^{n(n+1)/2} (-z; q)_n}{(-xyq; q)_n (q; q)_n}.$$

We do *not* get a new summation formula from combining (3.16) with Corollary 13, but we do get that if $a(q)$ and $b(q)$ are as defined in Corollary 13, then $a(q) = a(-q)$, and $b(q) = b(-q)$. This was initially not obvious, but actually follows immediately upon replacing j with $n-j$ in the equations defining $a(q)$ and $b(q)$.

Corollary 18. For $|q| < 1$ and integral $m \geq 0$,

$$(3.17) \quad \sum_{n=0}^{\infty} \frac{(-q^2; q^2)_{m+n-1} q^n}{(q; q^2)_{m+n} (q^2; q^2)_n} = \frac{(-1)^{m-1} (-q; q)_\infty}{q^{m(m-1)} (q; q)_\infty} \\ \times \left[a_m(q)(q^2, q^{10}, q^{12}; q^{12})_\infty - b_m(q)(q^6, q^6, q^{12}; q^{12})_\infty \right],$$

where $a_0(q) = 0$, $b_0(q) = 1$, and for $m \geq 1$,

$$a_m(q) = \sum_{n, j, l \geq 0} (-1)^{j+l+n} q^{n^2+l^2} \begin{bmatrix} m-1-n+j \\ j \end{bmatrix}_{q^2} \\ \times \begin{bmatrix} m-1-j-l \\ n-j-l \end{bmatrix}_{q^2} \begin{bmatrix} m-1-n \\ l \end{bmatrix}_{q^2}, \\ b_m(q) = \sum_{n, j, l \geq 0} (-1)^{j+l+n} q^{n^2+2n+l^2} \begin{bmatrix} m-2-n+j \\ j \end{bmatrix}_{q^2} \\ \times \begin{bmatrix} m-2-j-l \\ n-j-l \end{bmatrix}_{q^2} \begin{bmatrix} m-2-n \\ l \end{bmatrix}_{q^2}.$$

Proof. In (3.16), replace q with q^2 , set $x = q^{2m-1}$, $y = -1$ and $z = q$. Combine the resulting identity with (3.12), and finally replace q with $-q$. \square

3.3. Ramanujan's Transformation Again. If a factor of $1/(1-y)$ is cancelled in (2.27) and then y is replaced with $-xy$ and the resulting equation rearranged, we get

$$(3.18) \quad \sum_{n \geq 0} \frac{(zx)^n q^{n^2} (qxy/z; q)_n}{(-xq; q)_n (-xyq; q)_n (q; q)_n} = \frac{1}{(-xq; q)_\infty} \sum_{n \geq 0} \frac{x^n q^{n(n+1)/2} (-z; q)_n}{(-xyq; q)_n (q; q)_n}.$$

Corollary 19. For $|q| < 1$ and integral $m \geq 0$,

$$(3.19) \quad \sum_{n=0}^{\infty} \frac{(-q^2; q^2)_{m+n-1} q^{2n^2+2mn}}{(q^2; q^4)_{m+n} (q^2; q^2)_n} = \frac{(-1)^{m-1} (-q^2; q^2)_{\infty}}{q^{m(m-1)} (q^2; q^2)_{\infty}} \\ \times \left[a_m(q) (q^2, q^{10}, q^{12}; q^{12})_{\infty} - b_m(q) (q^6, q^6, q^{12}; q^{12})_{\infty} \right],$$

where $a_0(q) = 0$, $b_0(q) = 1$, and for $m \geq 1$,

$$a_m(q) = \sum_{n,j,l \geq 0} (-1)^j q^{n^2+l^2} \begin{bmatrix} m-1-n+j \\ j \end{bmatrix}_{q^2} \\ \times \begin{bmatrix} m-1-j-l \\ n-j-l \end{bmatrix}_{q^2} \begin{bmatrix} m-1-n \\ l \end{bmatrix}_{q^2}, \\ b_m(q) = \sum_{n,j,l \geq 0} (-1)^j q^{n^2+2n+l^2} \begin{bmatrix} m-2-n+j \\ j \end{bmatrix}_{q^2} \\ \times \begin{bmatrix} m-2-j-l \\ n-j-l \end{bmatrix}_{q^2} \begin{bmatrix} m-2-n \\ l \end{bmatrix}_{q^2}.$$

Proof. In (3.18), replace q with q^2 , set $x = q^{2m-1}$, $y = -1$, $z = q$ and combine with (3.12). \square

4. m -VERSIONS OF IDENTITIES WHEN m IS A NEGATIVE INTEGER

In [12] and [7], the authors state some m -versions of identities, where m is a negative integer. We are also able to easily derive a large number of similar identities using our continued fraction approach. We first prove two general transformation, the negative m versions of those in Theorems 2 and 4.

Theorem 5. Let $m \geq 1$ be a positive integer.

(i) Let $\phi(x, y, z, q)$ and $e_m(x, y, z, q)$ be as defined in Theorem 2. Then

$$(4.1) \quad \phi(xq^{-m}, y, z, q) \\ = e_{m+1}(xq^{-m}, y, z, q) \phi(x, y, z, q) + (zx - y) e_m(xq^{-m}, y, z, q) \phi(xq, y, z, q).$$

(ii) Let $\Phi(x, y, z, q)$ and $g_m(x, y, z, q)$ be as defined in Theorem 4. Then

$$(4.2) \quad \Phi(xq^{-m}, y, z, q) = \frac{q^{m(m-1)/2}}{x^m y^m (-1/xy; q)_m} \left[g_{m+1}(xq^{-m}, y, z, q) \Phi(x, y, z, q) \right. \\ \left. + \frac{zx - x^2 y q}{1 + xyq} g_m(xq^{-m}, y, z, q) \Phi(xq, y, z, q) \right].$$

Proof. The first transformation follows immediately upon replacing x with xq^{-m} in (2.10), while the second follows similarly upon making the same substitution in (3.8), and then employing the identity (see [8, I.8, page 351])

$$(-xyq^{1-m}; q)_m = (-1/xy; q)_m (xy)^m q^{m(m-1)/2}.$$

□

We now give some examples of negative m -versions of identities. The negative m version of the Rogers-Ramanujan identities stated by Ismail and Stanton (case (5.4e) in [12]) is an easy consequence of (4.1) (our statement of this result is slightly different).

Corollary 20. *For each positive integer m ,*

$$(4.3) \quad \sum_{n=0}^{\infty} \frac{q^{n^2-mn}}{(q; q)_n} = \frac{a_m(q)}{(q, q^4; q^5)} + \frac{b_m(q)}{(q^2, q^3; q^5)},$$

where

$$a_m(q) = \sum_n q^{n^2-mn} \begin{bmatrix} m-n \\ n \end{bmatrix},$$

$$b_m(q) = \sum_n q^{n^2-mn} \begin{bmatrix} m-1-n \\ n \end{bmatrix}.$$

Proof. In (4.1), set $z = 1/x$, let $x, y \rightarrow 0$, and use (2.12). □

Remark: Polynomial generalizations of negative m -versions of identities may also be easily derived, by replacing x with xq^{-m} in (2.9) and (3.7), and then specializing x, y and z as before, but we do not investigate that here. We next give a negative m -version of the Göllnitz-Gordon identities.

Corollary 21. *For $|q| < 1$ and integral $m \geq 1$,*

$$(4.4) \quad \sum_{n=0}^{\infty} \frac{(-q; q^2)_n q^{n^2-2mn}}{(q^2; q^2)_n} = \frac{a_m(q)}{(q^3, q^4, q^5; q^8)_{\infty}} + \frac{b_m(q)}{(q, q^4, q^7; q^8)_{\infty}},$$

where

$$a_m(q) = \sum_{n,l} q^{n^2+l^2-2mn} \begin{bmatrix} m-1-l \\ n \end{bmatrix}_{q^2} \begin{bmatrix} n \\ l \end{bmatrix}_{q^2},$$

$$b_m(q) = \sum_{n,l} q^{n^2+l^2-2mn} \begin{bmatrix} m-l \\ n \end{bmatrix}_{q^2} \begin{bmatrix} n \\ l \end{bmatrix}_{q^2}.$$

Proof. In (4.1), replace q with q^2 , set $x = 1/q$, $y = 0$, $z = q$ and use (2.17). □

We next give a negative m -version of the identity in Corollary 3.

Corollary 22. *For $|q| < 1$ and integral $m \geq 1$,*

$$(4.5) \quad \sum_{n=0}^{\infty} \frac{q^{n^2-2mn}}{(q^4, q^4)_n} = \frac{a_m(q)}{(q^2, q^3; q^5)_{\infty} (-q^2; q^2)_{\infty}} + \frac{b_m(q)}{(q, q^4; q^5)_{\infty} (-q^2; q^2)_{\infty}},$$

where

$$a_m(q) = \sum_{n,j} q^{n^2-2mn} (-1)^j \begin{bmatrix} m-1-j \\ n \end{bmatrix}_{q^2} \begin{bmatrix} n+j \\ j \end{bmatrix}_{q^2},$$

$$b_m(q) = \sum_{n,j} q^{n^2-2mn} (-1)^j \begin{bmatrix} m-j \\ n \end{bmatrix}_{q^2} \begin{bmatrix} n+j \\ j \end{bmatrix}_{q^2}.$$

Proof. In (4.1), replace q with q^2 , set $x = 1/q$, $y = -1$, $z = 0$ and use the identities at (2.15). \square

Note that Watson's identity (2.18) does not give anything very interesting when applied to negative m -versions of identities for which $z = 1/x$, since replacing x with xq^{-m} and then setting $z = 1/x$ causes the product $(zxq; q)_\infty$ on the right at (2.18) to vanish. However, we do get a new identity when $z \neq 1/x$. We give a negative m -version of Corollary 7 as an example.

Corollary 23. For $|q| < 1$ and integral $m \geq 1$,

$$(4.6) \quad \sum_{n=0}^{\infty} \frac{q^{3n^2-2mn} (-1)^n}{(-q^{1-2m}; q^2)_n (q^4; q^4)_n}$$

$$= \frac{q^{m^2}}{(-q, q^2)_m} \left[\frac{a_m(q)}{(q^2, q^3; q^5)_\infty (-q; q)_\infty} + \frac{b_m(q)}{(q, q^4; q^5)_\infty (-q; q)_\infty} \right],$$

where

$$a_m(q) = \sum_{n,j} q^{n^2-2mn} (-1)^j \begin{bmatrix} m-1-j \\ n \end{bmatrix}_{q^2} \begin{bmatrix} n+j \\ j \end{bmatrix}_{q^2},$$

$$b_m(q) = \sum_{n,j} q^{n^2-2mn} (-1)^j \begin{bmatrix} m-j \\ n \end{bmatrix}_{q^2} \begin{bmatrix} n+j \\ j \end{bmatrix}_{q^2}.$$

Proof. In (2.18), replace q with q^2 and x with xq^{-2m} , and then set $x = 1/q$, $y = -1$ and $z = 0$. Combine the resulting identity with (4.5), and now (4.6) follows after a little manipulation. \square

Heine's transformation (2.23) and Ramanujan's transformation (2.27) may be similarly used in conjunction with some existing negative m -versions of identities to produce new negative m -versions.

Corollary 24. For $|q| < 1$ and integral $m \geq 1$,

$$(4.7) \quad \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(-q^{1-2m}; q^2)_n (q^2; q^2)_n}$$

$$= \frac{q^{m^2}}{(-q; q^2)_m} \left[\frac{a_m(q)}{(q^2, q^3; q^5)_\infty (-q; q^2)_\infty} + \frac{b_m(q)}{(q, q^4; q^5)_\infty (-q; q^2)_\infty} \right],$$

where $a_0(q) = 0$, $b_0(q) = 1$, and for $m \geq 1$,

$$a_m(q) = \sum_{n,j} q^{n^2-2mn} (-1)^j \begin{bmatrix} m-1-j \\ n \end{bmatrix}_{q^2} \begin{bmatrix} n+j \\ j \end{bmatrix}_{q^2},$$

$$b_m(q) = \sum_{n,j} q^{n^2-2mn} (-1)^j \begin{bmatrix} m-j \\ n \end{bmatrix}_{q^2} \begin{bmatrix} n+j \\ j \end{bmatrix}_{q^2}.$$

Proof. In (2.23), replace q with q^2 and x with xq^{-2m} , set $x = 1/q$ and $y = -1$, let $z \rightarrow 0$ and combine with (4.5). \square

Corollary 25. For $|q| < 1$ and integral $m \geq 1$,

$$(4.8) \quad \sum_{n=0}^{\infty} \frac{q^{2n^2-2mn}}{(-q^{1-2m}; q^2)_n (q^2; q^2)_n}$$

$$= \frac{q^{m^2}}{(-q; q^2)_m} \left[\frac{a_m(q)}{(q^3, q^4, q^5; q^8)_{\infty} (-q; q^2)_{\infty}} + \frac{b_m(q)}{(q, q^4, q^7; q^8)_{\infty} (-q; q^2)_{\infty}} \right],$$

where

$$a_m(q) = \sum_{n,l} q^{n^2+l^2-2mn} \begin{bmatrix} m-1-l \\ n \end{bmatrix}_{q^2} \begin{bmatrix} n \\ l \end{bmatrix}_{q^2},$$

$$b_m(q) = \sum_{n,l} q^{n^2+l^2-2mn} \begin{bmatrix} m-l \\ n \end{bmatrix}_{q^2} \begin{bmatrix} n \\ l \end{bmatrix}_{q^2}.$$

Proof. In (2.27), replace q with q^2 and x with xq^{-2m} , set $x = 1/q$, $y = 0$, $z = q$ and combine with (4.4). \square

All of the negative m -versions so far in this section have followed from either (4.1), or from (4.1) in conjunction with one of the transformations at (2.18), (2.23) or (2.27). We now consider (4.2). We limit this consideration to a negative m -version of (3.9), although we could have derived negative m -versions of (3.11) and (3.12), and also derived yet further negative m -versions by then applying (2.18), (2.23) or (2.27) to each of those identities.

Corollary 26. For $|q| < 1$ and integral $m \geq 1$,

$$(4.9) \quad \sum_{n=0}^{\infty} \frac{q^{n^2-2mn}}{(q^{1-2m}; q^2)_n (q^2; q^2)_n} =$$

$$\frac{q^{m^2} (-1)^m (-q; q^2)_{\infty}}{(q; q^2)_m (q^2; q^2)_{\infty}} \left[a_m(q) (q^4, q^{16}, q^{20}; q^{20})_{\infty} + b_m(q) (q^8, q^{12}, q^{20}; q^{20})_{\infty} \right],$$

where

$$a_m(q) = \sum_{n,j} q^{n^2-2mn} (-1)^j \begin{bmatrix} m-1-n+j \\ j \end{bmatrix}_{q^2} \begin{bmatrix} m-1-j \\ n-j \end{bmatrix}_{q^2},$$

$$b_m(q) = \sum_{n,j} q^{n^2-2mn} (-1)^j \begin{bmatrix} m-n+j \\ j \end{bmatrix}_{q^2} \begin{bmatrix} m-j \\ n-j \end{bmatrix}_{q^2}.$$

Proof. In (4.2) replace q with q^2 and x with xq^{-2m} , set $x = 1/q$, $y = -1$, $z = 0$ and use the identities at (3.10). \square

5. CONCLUDING REMARKS

The authors in [3] give a polynomial generalization of (1.1) and give similar identities in [4]. As we have already noted, similar generalizations of many of the identities in the present paper could also have been stated. However, we do not state them explicitly here - the interested reader may easily derive them, if desired, by specializing x , y and z in (2.9) or (3.7) in the same way that they were specialized in (2.8) or (3.5) to produce the identities in the various corollaries.

We point out that the methods outlined in the present paper do not give all of the m -versions arising from pairs of identities of Rogers-Ramanujan type that have been derived elsewhere (for example, they do not lead to a proof of m -version in Theorem 2.7 in [9]). Our methods are also restricted to m -versions of *pairs* of Slater-type identities (since the recurrence relations for the partial numerators and denominators of continued fractions are *three-term* recurrences), so that they will not lead to m -versions of *triples* of Slater-type identities, such as have been given in [9] and elsewhere.

It may also be the case that there are other transformations for basic hypergeometric series that may be used to derive further m -versions, in ways similar to how those of Watson, Heine and Ramanujan were used in the present paper.

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