2006

Further Combinatorial Identities Deriving from the n-th Power of a 2 X 2 Matrix

James McLaughlin
West Chester University of Pennsylvania, jmclaughlin2@wcupa.edu

Nancy Wyshinski
Trinity College

FURTHER COMBINATORIAL IDENTITIES DERIVING FROM THE n-TH POWER OF A 2×2 MATRIX

J. MC LAUGHLIN AND NANCY J. WYSHINSKI

Abstract. In this paper we use a formula for the n-th power of a 2×2 matrix A (in terms of the entries in A) to derive various combinatorial identities. Three examples of our results follow.

1) We show that if m and n are positive integers and $s \in \{0, 1, 2, \dots, \lfloor (mn-1)/2 \rfloor\}$, then

$$\sum_{i,j,k,t} 2^{1+2t-mn+n} \frac{(-1)^{nk+i(n+1)}}{1+\delta_{(m-1)/2,\, i+k}} \binom{m-1-i}{i} \binom{m-1-2i}{k} \times \binom{n(m-1-2(i+k))}{2j} \binom{j}{t-n(i+k)} \binom{n-1-s+t}{s-t} = \binom{mn-1-s}{s}.$$

2) The generalized Fibonacci polynomial $f_m(x,s)$ can be expressed as

$$f_m(x,s) = \sum_{k=0}^{\lfloor (m-1)/2 \rfloor} \binom{m-k-1}{k} x^{m-2k-1} s^k.$$

We prove that the following functional equation holds:

$$f_{mn}(x,s) = f_m(x,s) \times f_n\left(f_{m+1}(x,s) + s f_{m-1}(x,s),\, -(-s)^m \right).$$

3) If an arithmetical function f is multiplicative and for each prime p there is a complex number $g(p)$ such that

$$f(p^{n+1}) = f(p) f(p^n) - g(p) f(p^{n-1}), \qquad n \geq 1,$$

then f is said to be *specially multiplicative*. We give another derivation of the following formula for a specially multiplicative function f evaluated at a prime power:

$$f(p^k) = \sum_{j=0}^{\lfloor k/2 \rfloor} (-1)^j \binom{k-j}{j} f(p)^{k-2j} g(p)^j.$$

We also prove various other combinatorial identities.

Date: October 11th, 2004.

1991 *Mathematics Subject Classification.* 05A19.

Key words and phrases. Combinatorial Identities.

1. Introduction

Throughout the paper, let I denote the 2×2-identity matrix and n an arbitrary positive integer. In [7], the first author proved the following theorem, which gives a formula for the n-th power of a 2×2 matrix in terms of its entries:

Theorem 1. *Let*

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

be an arbitrary 2×2 matrix and let $T = a+d$ denote its trace and $D = ad-bc$ its determinant. Let

$$y_n = \sum_{i=0}^{\lfloor n/2 \rfloor} \binom{n-i}{i} T^{n-2i}(-D)^i.$$

Then, for $n \geq 1$,

$$A^n = \begin{pmatrix} y_n - d\, y_{n-1} & b\, y_{n-1} \\ c\, y_{n-1} & y_n - a\, y_{n-1} \end{pmatrix}.$$

The proof used the fact that

$$y_{k+1} = (a+d)y_k + (bc-ad)y_{k-1}. \tag{1.1}$$

This theorem was then used to derive various binomial identities. As an example, we cite the following corollary.

Corollary 1. *Let n be a positive integer and let m be an integer with $0 \leq m \leq 2n$. Then for $-n \leq w \leq n$,*

$$\sum_{k=0}^{n-1} \binom{n-1-k}{k}\binom{n}{w+k}\binom{k+w}{m-k-w}(-1)^k =$$
$$\sum_{k=-2w-n+m+1}^{m-w} \binom{n}{k+w}\binom{n}{n+k+w-m}\binom{k+n+2w-m-1}{k}(-1)^k.$$

In this present paper we use Theorem 1 to derive some further identities.

2. A Binomial Identity deriving from $(A^m)^n = A^{mn}$

We use the trivial identity $(A^m)^n = A^{mn}$ to prove the following theorem.

Theorem 2. *Let m and n be positive integers and let $s \in \{0, 1, 2, \ldots, \lfloor (mn-1)/2 \rfloor\}$. Then*

$$\sum_{i,j,k,t} 2^{1+2t-mn+n} \frac{(-1)^{nk+i(n+1)}}{1+\delta_{(m-1)/2,\, i+k}} \binom{m-1-i}{i}\binom{m-1-2i}{k} \times \tag{2.1}$$
$$\binom{n(m-1-2(i+k))}{2j}\binom{j}{t-n(i+k)}\binom{n-1-s+t}{s-t} = \binom{mn-1-s}{s},$$

where i, j, k and t run through integral values which keep all binomial entries in (2.1) *non-negative, and*

$$\delta_{p,\,q} = \begin{cases} 1, & p = q, \\ 0, & p \neq q. \end{cases}$$

Proof. Let

$$A = \begin{pmatrix} 1 & 1 \\ x & 0 \end{pmatrix}.$$

From Theorem 1 and (1.1) we have that

$$A^n = \begin{pmatrix} y_n & y_{n-1} \\ x\, y_{n-1} & y_n - y_{n-1} \end{pmatrix} = \begin{pmatrix} y_n & y_{n-1} \\ x\, y_{n-1} & x\, y_{n-2} \end{pmatrix}, \tag{2.2}$$

with

$$y_k = \sum_{i=0}^{\lfloor k/2 \rfloor} \binom{k-i}{i} x^i = y_{k-1} + x\, y_{k-2}. \tag{2.3}$$

Let T_n denote the trace of A^n and D_n the determinant of A^n (so $D_n = (-x)^n$). From (2.2) we have that

$$T_n = y_n + x\, y_{n-2}. \tag{2.4}$$

Thus the sequence $\{T_n\}$ satisfies the same recurrence relation as the sequence $\{y_n\}$, namely

$$T_{n+1} = T_n + xT_{n-1}.$$

This leads to the explicit formula

$$\begin{aligned} T_n &= \left(\frac{1+\sqrt{1+4x}}{2}\right)^n + \left(\frac{1-\sqrt{1+4x}}{2}\right)^n \\ &= \frac{1}{2^{n+1}} \sum_{j=0}^{\lfloor n/2 \rfloor} \sum_{k=j}^{\lfloor n/2 \rfloor} \binom{n}{2k}\binom{k}{j} 4^j x^j. \end{aligned} \tag{2.5}$$

After some straightforward but tedious calculations, we derive from the first of these equalities, for integral $r \geq 0$, that

(2.6)
$$T_n^r = \sum_{s=0}^{\lfloor nr/2 \rfloor} \sum_{k=0}^{\lfloor r/2 \rfloor} \sum_{i=0}^{\lfloor n(r-2k)/2 \rfloor} \binom{r}{k} \binom{n(r-2k)}{2i} \binom{i}{s-nk} 2^{1+2t-rn} \frac{(-1)^{nk}}{1+\delta_{r/2,\,k}} x^s.$$

As usual,

$$\delta_{p,\,q} = \begin{cases} 1, & p = q, \\ 0, & p \neq q. \end{cases}$$

For integral $j \geq 0$ define

$$y_j^{(n)} = \sum_{i=0}^{\lfloor j/2 \rfloor} \binom{j-i}{i} T_n^{j-2i} (-D_n)^i. \tag{2.7}$$

Then Theorem 1 and the trivial identity $A^{mn}=(A^n)^m$ give that

$$\begin{aligned}\begin{pmatrix} y_{mn} & y_{mn-1}\\ x\,y_{mn-1} & xy_{mn-2}\end{pmatrix} &= \begin{pmatrix} y_{n} & y_{n-1}\\ x\,y_{n-1} & xy_{n-2}\end{pmatrix}^m\\ &= \begin{pmatrix} y_m^{(n)}-xy_{n-2}y_{m-1}^{(n)} & y_{n-1}y_{m-1}^{(n)}\\ x\,y_{n-1}y_{m-1}^{(n)} & y_m^{(n)}-y_{n-1}y_{m-1}^{(n)}\end{pmatrix}.\end{aligned}$$

If we compare $(1,2)$ entries of the first and last matrices, we have that

$$y_{mn-1}=y_{n-1}y_{m-1}^{(n)}.$$

Upon combining (2.7), (2.6) and (2.3), we get that

$$\begin{aligned}&\sum_{s=0}^{\lfloor (mn-1)/2\rfloor}\binom{mn-1-s}{s}x^s\\ &=\sum_{s=0}^{\lfloor (mn-1)/2\rfloor}\sum_{i,j,k,t}2^{1+2t-mn+n}\frac{(-1)^{nk+i(n+1)}}{1+\delta_{(m-1)/2,\,i+k}}\binom{m-1-i}{i}\binom{m-1-2i}{k}\\ &\qquad\times\binom{n(m-1-2(i+k))}{2j}\binom{j}{t-n(i+k)}\binom{n-1-s+t}{s-t}x^s.\end{aligned}$$

Here i, j, k, and t run through all sets of integers which keep all binomial entries non-negative. The result now follows upon comparing coefficients of like powers of x. ☐

Upon comparing like powers of x on each side of (2.4), using (2.3) and (2.5), we get the following.

Corollary 2. *Let n be a positive integer. Then for each integer s, $0\le s\le \lfloor n/2\rfloor$,*

$$\frac{1}{2^{n-2s-1}}\sum_{j=s}^{\lfloor n/2\rfloor}\binom{n}{2j}\binom{j}{s}=\frac{n}{n-s}\binom{n-s}{s}.$$

This identity is also found in [1] (page 442) and [3] (formula 3.120).

3. A Proof of an Identity for Specially Multiplicative Functions

An arithmetical function f is said to be *multiplicative* if $f(1)=1$ and

$$f(mn)=f(m)f(n), \tag{3.1}$$

whenever $(m,n)=1$. If (3.1) holds for all m and n, then f is said to be *completely multiplicative*. A multiplicative function f is said to be *specially multiplicative* if there is a completely multiplicative function f_A such that

$$f(m)f(n)=\sum_{d|(m,n)} f\left(\frac{mn}{d^2}\right)f_A(d)$$

for all m and n. An alternative characterization of specially multiplicative functions is given below (see [5], for example):

If f is multiplicative and for each prime p there is a complex number $g(p)$ such that

$$f(p^{n+1}) = f(p)f(p^n) - g(p)f(p^{n-1}), \qquad n \geq 1, \tag{3.2}$$

then f is specially multiplicative. (In this case, $f_A(p) = g(p)$, for all primes p).

We give an alternative proof of the following known result (also see [5], for example).

Proposition 1. *Let f and g be as at* (3.2). *Then for $k \geq 0$ and all primes p,*

$$f(p^k) = \sum_{j=0}^{\lfloor k/2 \rfloor} (-1)^j \binom{k-j}{j} f(p)^{k-2j} g(p)^j.$$

Proof. Clearly we can assume $k \geq 3$. Equation 3.2 implies that

(3.3)

$$\begin{aligned}
\begin{pmatrix} f(p^k) & f(p^{k-1}) \\ f(p^{k-1}) & f(p^{k-2}) \end{pmatrix} &= \begin{pmatrix} f(p^{k-1}) & f(p^{k-2}) \\ f(p^{k-2}) & f(p^{k-3}) \end{pmatrix} \begin{pmatrix} f(p) & 1 \\ -g(p) & 0 \end{pmatrix} \\
&= \begin{pmatrix} f(p^2) & f(p) \\ f(p) & 1 \end{pmatrix} \begin{pmatrix} f(p) & 1 \\ -g(p) & 0 \end{pmatrix}^{k-2} \\
&= \begin{pmatrix} f(p)^2 - g(p) & f(p) \\ f(p) & 1 \end{pmatrix} \begin{pmatrix} f(p) & 1 \\ -g(p) & 0 \end{pmatrix}^{k-2} \\
&= \left(\begin{pmatrix} 0 & 0 \\ 1 + g(p) & 0 \end{pmatrix} + \begin{pmatrix} f(p) & 1 \\ -g(p) & 0 \end{pmatrix} \right) \begin{pmatrix} f(p) & 1 \\ -g(p) & 0 \end{pmatrix}^{k-1}.
\end{aligned}$$

The result now follows immediately from Theorem 1, upon comparing $(1, 1)$ entries on each side. $\square$

Remark: The Ramanujan τ function is specially multiplicative with $g(p) = p^{11}$. We note in passing that the τ Conjecture for p prime, namely that $|\tau(p)| < 2p^{11/2}$, is equivalent to the conjecture that $\lim_{k\to\infty} \tau(p^k)/\tau(p^{k-1})$ does not exist. This follows from (3.3), the correspondence between matrices and continued fractions and Worpitzky's Theorem for continued fractions.

4. A Recurrence Formula for the Generalized Fibonacci Polynomials

The Fibonacci polynomials $\{f_m(x,s)\}_{m=0}^{\infty}$ are defined by $f_0(x,s) = 0$, $f_1(x,s) = 1$ and $f_{n+1}(x,s) = xf_n(x,s) + sf_{n-1}(x,s)$, for $n \geq 1$. They are given explicitly by the formula

$$f_m(x,s) = \sum_{k=0}^{\lfloor (m-1)/2 \rfloor} \binom{m-k-1}{k} x^{m-2k-1} s^k.$$

It is clear from Theorem 1 that the $f_n(x,s)$ satisfy

$$\begin{aligned}\begin{pmatrix} x & 1 \\ s & 0 \end{pmatrix}^m &= \begin{pmatrix} f_{m+1}(x,s) & f_m(x,s) \\ sf_m(x,s) & f_{m+1}(x,s) - xf_m(x,s) \end{pmatrix} \\ &= \begin{pmatrix} f_{m+1}(x,s) & f_m(x,s) \\ sf_m(x,s) & sf_{m-1}(x,s) \end{pmatrix}.\end{aligned}$$

We can now use the trivial identity $A^{mn} = (A^m)^n$ applied to the matrix $\begin{pmatrix} x & 1 \\ s & 0 \end{pmatrix}$, together with Theorem 1 applied to the $(1,2)$-entries on each side to get the following functional equation for the Fibonacci polynomials.

Corollary 3. *Let $f_i(x,s)$ denote the i-th Fibonacci polynomial and let m and n be positive integers. Then*

$$\begin{aligned}&f_{mn}(x,s) \\ &= f_m(x,s) \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-k-1}{k} \left[f_{m+1}(x,s) + sf_{m-1}(x,s) \right]^{n-2k-1} (-(-s)^m)^k \\ &= f_m(x,s) \times f_n \left(f_{m+1}(x,s) + sf_{m-1}(x,s), \, -(-s)^m \right).\end{aligned}$$

5. A Polynomial Identity of Bhatwadekar and Roy

In [8] Sury gave a proof of the following polynomial identity, which he attributes to Bhatwadekar and Roy [2]:

Corollary 4. *For every positive integer n and all x,*

$$\sum_{i=0}^{\lfloor n/2 \rfloor} (-1)^i \binom{n-i}{i} x^i (1+x)^{n-2i} = 1 + x + \cdots + x^n.$$

Proof. Clearly we can assume $n \geq 2$. One easily checks by induction that, for $n \geq 2$,

$$\frac{1}{1-x} \begin{pmatrix} 1-x^{n+1} & 1-x^n \\ -x(1-x^n) & -x(1-x^{n-1}) \end{pmatrix} = \begin{pmatrix} 1+x & 1 \\ -x & 0 \end{pmatrix}^n.$$

The result is now immediate from Theorem 1. □

6. Other Elementary Identities

If we replace n by $n+1$ in Equation 2.2 and take the determinant of the first and last matrices, we get

$$(-x)^{n+1} = x(y_{n+1}y_{n-1} - y_n^2).$$

Upon comparing coefficients of x^s, for $0 \leq s \leq n-1$ on each side, we get the following identity.

Corollary 5. *Let n be a positive integer. If s is an integer, $0 \leq s \leq n-1$, then*

$$\sum_{j\geq 0}\binom{n-s+j}{s-j}\binom{n-j}{j} = \sum_{j\geq 0}\binom{n+1-s+j}{s-j}\binom{n-1-j}{j}. \tag{6.1}$$

Once again we start with the matrix $A = \left(\begin{smallmatrix}1 & 1\\ x & 0\end{smallmatrix}\right)$ and then consider the identity $A^{mn} = (A^m)^n = (A^n)^m$ for small values of m.

Corollary 6. *Let n be a positive integer and let s be an integer, $0 \leq s \leq n-1$. Then*

$$\begin{aligned}(6.2)\quad &\sum_{i\geq 0}\binom{n-i-1}{i}\binom{n-2i-1}{s-2i}2^{s-2i}(-1)^i \\ &= \sum_{i=0}^{\lfloor n/2\rfloor}\frac{n}{n-i}\binom{n-i}{i}\binom{n+i-s-1}{s-i} = \binom{2n-s-1}{s}.\end{aligned}$$

Proof. With A as defined above, we have

$$A^2 = \begin{pmatrix} x+1 & 1\\ x & x\end{pmatrix}.$$

If we compare the $(1,2)$ entries of A^{2n} and $(A^2)^n$, using Theorem 1, we get that

$$\begin{aligned}\sum_{s=0}^{n-1}\binom{2n-s-1}{s}x^s &= \sum_{i=0}^{\lfloor\frac{n-1}{2}\rfloor}\binom{n-i-1}{i}(2x+1)^{n-2i-1}(-x^2)^i \\ &= \sum_{i=0}^{\lfloor\frac{n-1}{2}\rfloor}\sum_{j=0}^{n-2i-1}\binom{n-i-1}{i}\binom{n-2i-1}{j}2^j(-1)^i x^{2i+j} \\ &= \sum_{s=0}^{n-1}\sum_{i\geq 0}\binom{n-i-1}{i}\binom{n-2i-1}{s-2i}2^{s-2i}(-1)^i x^s.\end{aligned}$$

The equality of the first and third terms in (6.2) follows on comparing powers of x. On the other hand, Theorem 1 also gives that

$$\begin{aligned}A^{2n} = (A^n)^2 &= \begin{pmatrix} y_n & y_{n-1}\\ x\,y_{n-1} & y_n - y_{n-1}\end{pmatrix}^2 = \begin{pmatrix} y_n & y_{n-1}\\ x\,y_{n-1} & x\,y_{n-2}\end{pmatrix}^2 \\ &= \begin{pmatrix} y_n^2 + xy_{n-1}^2 & y_{n-1}(y_n + x\,y_{n-2})\\ x\,y_{n-1}(y_n + x\,y_{n-2}) & x(y_{n-1}^2 + x\,y_{n-2}^2)\end{pmatrix},\end{aligned}$$

where y_k is as at (2.3). It is easy to show that

$$y_n + x\,y_{n-2} = \sum_{i=0}^{\lfloor n/2\rfloor}\frac{n}{n-i}\binom{n-i}{i}x^i. \tag{6.3}$$

If we compare the $(1,2)$ entries of A^{2n} and $(A^n)^2$ using (6.3) and Theorem 1, then

$$\sum_{s=0}^{n-1}\binom{2n-s-1}{s}x^s = \sum_{k=0}^{\lfloor\frac{n-1}{2}\rfloor}\sum_{i=0}^{\lfloor n/2\rfloor}\frac{n}{n-i}\binom{n-i}{i}\binom{n-k-1}{k}x^{i+k}$$
$$= \sum_{s=0}^{n-1}\sum_{i=0}^{\lfloor n/2\rfloor}\frac{n}{n-i}\binom{n-i}{i}\binom{n+i-s-1}{s-i}x^s.$$

The equality of the second and third terms in (6.2) now follows. □

A similar consideration of A^{3n} and $(A^3)^n$ gives the following identity.

Corollary 7. *Let n be a positive integer and s an integer such that $0 \le s \le \lfloor(3n-1)/2\rfloor$. Then*
(6.4)
$$\sum_{i=0}^{\lfloor n/2\rfloor} 3^{s-1-3i}\binom{n-i-1}{i}\left(\binom{n-2i}{s-3i-1}+3\binom{n-2i}{s-3i}\right) = \binom{3n-s-1}{s}.$$

Proof. Since
$$A^3 = \begin{pmatrix} 2x+1 & x+1 \\ x^2+x & x \end{pmatrix},$$
comparing the $(1,2)$ entries of A^{3n} and $(A^3)^n$, using Theorem 1, gives

$$\sum_{s=0}^{\lfloor\frac{3n-1}{2}\rfloor}\binom{3n-s-1}{s}x^s = (x+1)\sum_{i=0}^{\lfloor\frac{n-1}{2}\rfloor}\binom{n-i-1}{i}(3x+1)^{n-2i-1}x^{3i}. \tag{6.5}$$

The results follows, after a little simplification, upon comparing coefficients of like powers of x on each side of (6.5). □

More generally, one can use the identity $A^{m+n} = A^m A^n$ together with Theorem 1 to compare the $(1,1)$ entries on each side to get (again using the notation from (2.3)) that

$$y_{m+n} = y_m y_n + y_{m-1}(x\, y_{n-1}).$$

Upon collecting like powers of x and equating coefficients on each side, we get the following identity.

Corollary 8. *Let m and n be a positive integer and s an integer such that $0 \le s \le \lfloor(m+n)/2\rfloor$. Then*
(6.6)
$$\sum_{i\ge 0}\binom{m-i}{i}\binom{n-s+i}{s-i}+\binom{m-i-1}{i}\binom{n-s+i}{s-i-1} = \binom{m+n-s}{s}.$$

7. Concluding Remarks

Some other interesting consequences follow readily from Theorem 1. We consider two more.

If we let $A = \left(\begin{smallmatrix} x & 0 \\ 0 & y \end{smallmatrix}\right)$, then Waring's formula

$$x^n + y^n = \sum_{j=0}^{\lfloor n/2 \rfloor} \frac{n}{n-j} \binom{n-j}{j} (x+y)^{n-2j} (-x\,y)^j$$

can be derived easily by considering the trace of A^n.

If we set $A = \left(\begin{smallmatrix} x & 1 \\ 1 & 0 \end{smallmatrix}\right)$, then Theorem 1 and the correspondence between continued fractions and matrices give that, for $x > 0$,

$$\lim_{n\to\infty} \frac{\displaystyle\sum_{j=0}^{\lfloor n/2 \rfloor} \binom{n-j}{j} x^{n-2j}}{\displaystyle\sum_{j=0}^{\lfloor (n-1)/2 \rfloor} \binom{n-1-j}{j} x^{n-1-2j}} = \frac{2}{\sqrt{x^2+4}-x}.$$

References

[1] Berndt, Bruce C.; Evans, Ronald J.; Williams, Kenneth S. *Gauss and Jacobi sums.* Canadian Mathematical Society Series of Monographs and Advanced Texts. A Wiley-Interscience Publication. John Wiley & Sons, Inc., New York, 1998. xii+583 pp.

[2] S. M. Bhatwadekar, A. Roy, *Some results on embedding of a line in* 3*-space.* J. Algebra **142** (1991), no. 1, 101–109.

[3] Gould, Henry W. *Combinatorial identities. A standardized set of tables listing 500 binomial coefficient summations.* Henry W. Gould, Morgantown, W.Va., 1972. viii+106 pp.

[4] H. W. Gould, *A history of the Fibonacci Q-matrix and a higher-dimensional problem.* Fibonacci Quart. **19** (1981), no. 3, 250–257.

[5] P. Haukkanen, *Some characterizations of specially multiplicative functions.* Int. J. Math. Math. Sci. **2003**, no. 37, 2335–2344.

[6] R. C. Johnson, *Matrix methods for Fibonacci and related sequences.* http:// maths.dur.ac.uk/ dma0rcj/PED/fib.pdf, (August 2003).

[7] James Mc Laughlin, *Combinatorial Identities Deriving from the n-th Power of a* 2×2 *Matrix.* – submitted

[8] B. Sury, *A curious polynomial identity.* Nieuw Arch. Wisk. (4) **11** (1993), no. 2, 93–96.

[9] Kenneth S. Williams, *The nth Power of a* 2×2 *Matrix (in Notes).* Mathematics Magazine, Vol. **65**, No. 5. (Dec., 1992), p. 336.

Mathematics Department, Trinity College, 300 Summit Street, Hartford, CT 06106-3100

E-mail address: james.mclaughlin@trincoll.edu

Mathematics Department, Trinity College, 300 Summit Street, Hartford, CT 06106-3100

E-mail address: nancy.wyshinski@trincoll.edu