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# The Convergence behavior of q-Continued Fractions on the Unit Circle

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# THE CONVERGENCE BEHAVIOR OF $q$-CONTINUED FRACTIONS ON THE UNIT CIRCLE

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ABSTRACT. In a previous paper, we showed the existence of an uncountable set of points on the unit circle at which the Rogers-Ramanujan continued fraction does not converge to a finite value.

In this present paper, we generalise this result to a wider class of $q$-continued fractions, a class which includes the Rogers-Ramanujan continued fraction and the three Ramanujan-Selberg continued fractions. We show, for each $q$-continued fraction, $G(q)$, in this class, that there is an uncountable set of points, $Y_G$, on the unit circle such that if $y \in Y_G$ then $G(y)$ does not converge to a finite value.

We discuss the implications of our theorems for the convergence of other $q$-continued fractions, for example the Göllnitz-Gordon continued fraction, on the unit circle.

## 1. INTRODUCTION

In a previous paper [2] we studied the convergence behavior of the celebrated Rogers–Ramanujan continued fraction, $R(q)$, which is defined for $|q| < 1$ by

$$R(q) := \frac{q^{1/5}}{1} \, {}_{+} \, \frac{q}{1} \, {}_{+} \, \frac{q^2}{1} \, {}_{+} \, \frac{q^3}{1} \, {}_{+} \, \cdots. \tag{1.1}$$

Put $K(q) = q^{1/5}/R(q)$.

Worpitzky's Theorem (see [6], pp. 35–36) gives that $R(q)$ converges to a value in $\hat{\mathbb{C}}$ for any $q$ inside the unit circle. Here $\hat{\mathbb{C}}$ denotes the extended complex plane.

**Theorem 1.** *(Worpitzky) Let the continued fraction $K_{n=1}^{\infty} a_n/1$ be such that $|a_n| \le 1/4$ for $n \ge 1$. Then $K_{n=1}^{\infty} a_n/1$ converges. All approximants of the continued fraction lie in the disc $|w| < 1/2$ and the value of the continued fraction is in the disk $|w| \le 1/2$.*

Outside the unit circle the odd and even parts of $K(q)$ tend to different limits. Suppose $|q| > 1$. For $n \ge 1$, define

$$K_n(q) := 1 + \frac{q}{1} \, {}_{+} \, \frac{q^2}{1} \, {}_{+} \, \frac{q^3}{1} \, {}_{+} \cdots {}_{+} \, \frac{q^n}{1}.$$

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Then

$$\lim_{j\to\infty} K_{2j+1}(q) = \frac{1}{K(-1/q)},$$

$$\lim_{j\to\infty} K_{2j}(q) = \frac{K(1/q^4)}{q}.$$

This was stated by Ramanujan without proof and proved by Andrews, Berndt, Jacobson and Lamphere in 1992 ([1]).

On the unit circle, there are two cases to consider. The easier of these is where $q$ is a root of unity, in which case $K(q)$ is periodic. Schur showed in [8] that if $q$ is a primitive $m$-th root of unity, where $m \equiv 0 \pmod 5$, then $K(q)$ diverges in the classical sense and if $q$ is a primitive $m$-th root of unity, $m \not\equiv 0 (\text{mod } 5)$, then $K(q)$ converges and

$$K(q) = \lambda\, q^{(1-\lambda\sigma m)/5} K(\lambda), \tag{1.2}$$

where $\lambda = \left(\frac{m}{5}\right)$ (the Legendre symbol) and $\sigma$ is the least positive residue of $m \,(\text{mod } 5)$. Note that $K(1) = \phi = (\sqrt{5}+1)/2$, and $K(-1) = 1/\phi$.

Remark: Schur's result was essentially proved by Ramanujan, probably earlier than Schur (see [7], p.383). However, he made a calculational error (see [4], p.56).

**Question:** Does the Rogers-Ramanujan continued fraction converge or diverge, in either the classical or general sense, at any point on the unit circle which is not a root of unity? This question had been open since Schur's 1917 paper until our paper [2]. In trying to prove convergence in the classical sense to a finite value, for example, one immediate difficulty is that Schur's theorem gives that there is a dense set on the unit circle at which $K(q)$ converges and another dense set at which $K(q)$ diverges. This immediately renders most of the usual convergence/divergence tests, which rely either on the partial quotients lying in certain subsets of $\mathbb{C}$ or on the absolute values of the partial quotients satisfying certain inequalities, useless.

To discuss this topic we use the following notation. Let the regular continued fraction expansion of any irrational $t \in (0,1)$ be denoted by $t = [0, e_1(t), e_2(t), \cdots]$. Let the $i$-th approximant of this continued fraction expansion be denoted by $c_i(t)/d_i(t)$. We will sometimes write $e_i$ for $e_i(t)$, $c_i$ for $c_i(t)$ etc, if there is no danger of ambiguity. Let $\phi = (\sqrt{5}+1)/2$. In [2], we proved the following theorem.

**Theorem 2.** *Let*

$$S = \{t \in (0,1) : e_{i+1}(t) \geq \phi^{d_i(t)} \textit{ infinitely often}\}. \tag{1.3}$$

*Then $S$ is an uncountable set of measure zero and, if $t \in S$ and $y = \exp(2\pi i t)$, then the Rogers-Ramanujan continued fraction does not converge to a finite value at $y$.*

We were also able to give explicit examples of points $y$ on the unit circle at which $K(y)$ did not converge to a finite value.

**Corollary 1.** *Let $t$ be the number with continued fraction expansion equal $[0, e_1, e_2, \cdots]$, where $e_i$ is the integer consisting of a tower of $i$ twos with an $i$ an top.*

$$t = [0, 2, 2^{2^2}, 2^{2^{2^3}}, \cdots] =$$
$$0.48484848484848484848484848484848484848484848484848484848484848484$$
$$84848484848484848484848484927788508311243752299231881201\,1\cdots$$

*If $y = \exp(2\pi i t)$ then $K(y)$ does not converge to a finite value.*

In [2] we were also able to show the existence of an uncountable set of points $G$ on the unit circle such that if $y \in G$, then $K(y)$ does not converge generally.

Remarks: In particular, this latter result shows the existence of an uncountable set of points $G$ on the unit circle at which the Rogers-Ramanujan continued fraction does not converge in the classical sense to values in $\hat{\mathbb{C}}$.

In [2] we also showed directly from the definition that $K(q)$ converges in the general sense at $5m$-th roots of unity, for $m$ a positive integer. This is in contrast to Schur's result which showed that $K(q)$ does not converge in the classical sense at $5m$-th roots of unity.

Remark: The divergence behavior of K(q) at $5m$-th roots of unity is an example of *Thiele oscillation* and convergence in the general sense can also be deduced from this Thiele oscillation. However, when we consider the case where $q$ is a point on the unit circle which is *not* a root of unity, the continued fraction is no longer periodic, or even limit-periodic, and general results about these types of continued fraction can no longer be applied.

In this paper we generalise Theorem 2 to a wider class of $q$-continued fractions, a class which includes the Rogers-Ramanujan continued fraction and the three Ramanujan-Selberg continued fractions (first studied by Ramanujan [7]) investigated by Zhang in [9].

$$S_1(q) := 1 + \frac{q}{1} + \frac{q+q^2}{1} + \frac{q^3}{1} + \frac{q^2+q^4}{1} + \cdots, \tag{1.4}$$

$$S_2(q) := 1 + \frac{q+q^2}{1} + \frac{q^4}{1} + \frac{q^3+q^6}{1} + \frac{q^8}{1} + \cdots, \tag{1.5}$$

and

$$S_3(q) := 1 + \frac{q+q^2}{1} + \frac{q^2+q^4}{1} + \frac{q^3+q^6}{1} + \frac{q^4+q^8}{1} + \cdots. \tag{1.6}$$

Thus, we show, for each of these continued fractions, the existence of an uncountable set of points on the unit circle at which the continued fraction does not converge to finite values.

## 2. Convergence Behavior of $q$-Continued Fractions on the Unit Circle

Let

$$G(q) := b_0(q) + K_{n=1}^{\infty} \frac{a_n(q)}{b_n(q)}, \tag{2.1}$$

where $a_n(q) \in \mathbb{Z}[q]$, for $n \geq 1$, $b_n(q) \in \mathbb{Z}[q]$, for $n \geq 0$, and $q$ is a complex variable.

For the remainder of the paper $P_n(q)/\, Q_n(q)$ denotes the $n$-th approximant of $G(q)$, $P_n/Q_n$ if there is no danger of ambiguity. It is well known (see [6], page 9) that, for $n \geq 1$,

$$P_n Q_{n-1} - P_{n-1} Q_n = (-1)^{n-1} \prod_{i=1}^{n} a_i. \tag{2.2}$$

For the continued fraction defined in (2.1) let

$$\chi_n(q) := \prod_{i=1}^{n} a_i(q). \tag{2.3}$$

We prove the following theorem.

**Theorem 3.** *Let $G(q)$ be as in (2.1). Suppose there exist constants $C_1$, $C_2 > 0$ and positive integers $j$ and $d$ such that if $m \equiv j (mod\ d)$ and $x_m$ is a primitive $m$-th root of unity, then there exists a positive integer $n(m)$, where $n(m) \to \infty$ as $m \to \infty$, such that*

$$|\chi_{n(m)}(x_m)| \geq C_1 \tag{2.4}$$

*and*

$$\max\{\ \left|Q_{n(m)}(x_m)\right|,\ \left|Q_{n(m)-1}(x_m)\right|\ \} \leq C_2. \tag{2.5}$$

*Then there is an uncountable set of points $Y_G$ on the unit circle such that if $y \in Y_G$ then $G(y)$ does not converge to a finite value.*

As a corollary to this theorem, we will show, for each of the continued fractions $K(q)$, $S_1(q)$, $S_2(q)$ and $S_3(q)$, that there exists an uncountable set of points on the unit circle at which the continued fraction does not converge to a finite value.

Let $G(q)$ be as defined in Equation 2.1 and suppose $G(q)$ converges at $q = y$ to $L \in \mathbb{C}$, so that $\lim_{n\to\infty} P_n(y)/Q_n(y) = L$.

$$\left|\frac{P_n(y)}{Q_n(y)} - \frac{P_{n-1}(y)}{Q_{n-1}(y)}\right| \leq \left|\frac{P_n(y)}{Q_n(y)} - L\right| + \left|\frac{P_{n-1}(y)}{Q_{n-1}(y)} - L\right|.$$

Thus

$$\lim_{n\to\infty}\left|\frac{P_n(y)}{Q_n(y)}-\frac{P_{n-1}(y)}{Q_{n-1}(y)}\right|=0. \tag{2.6}$$

We will exhibit an uncountable set of points for which (2.6) fails to hold, so that $G(q)$ does not converge to a finite value at any of these points.

We recall the notation introduced before the statement of Theorem 2. Let the regular continued fraction expansion of any irrational $t \in (0,1)$ be denoted by $t = [0, e_1(t), e_2(t), \cdots]$. Let the $i$-th approximant of this continued fraction expansion be denoted by $c_i(t)/d_i(t)$. We will sometimes write $e_i$ for $e_i(t)$, $c_i$ for $c_i(t)$ etc, if there is no danger of ambiguity. The key idea here is to construct real numbers $t$ in the interval $(0,1)$, each of which has the property that the sequence of approximants to its continued fraction expansion contains a subsequence of approximants which are "sufficiently close" to $t$ in a certain precise sense. Recall that it is possible to construct a real number $t$ for which the $m$-th approximant, $c_m/d_m$, in its continued fraction expansion, is as close to $t$ as desired by making the $m+1$-st partial quotient, $e_{m+1}$, sufficiently large. We then set $y := \exp(2\pi i t)$.

The purpose in constructing an irrational number $t$ of this form is to exert a certain amount of control over the absolute value of some of the terms in the sequence $\{Q_n(y)\}_{n=1}^{\infty}$. If $x_m := \exp(2\pi i c_m/d_m)$, then $y$ and $x_m$ are close enough to keep $\chi_{n(d_m)}(y)$ close to $\chi_{n(d_m)}(x_m)$ and $Q_{n(d_m)}(y)Q_{n(d_m)-1}(y)$ close to $Q_{n(d_m)}(x_m)Q_{n(d_m)-1}(x_m)$ for infinitely many $d_i$ in the sequence $\{d_m\}_{m=1}^{\infty}$. Equations (2.4) and (2.5) will then give that the sequence

$$\left\{\frac{P_n(y)}{Q_n(y)}-\frac{P_{n-1}(y)}{Q_{n-1}(y)}\right\}_{n=1}^{\infty}$$

contains an infinite subsequence which is bounded away from 0, contradicting the requirement at (2.6) and giving the result.

Before proving Theorem 3 it is necessary to prove some technical lemmas.

**Lemma 1.** *Let $G(q)$ be as in* (2.1). *There exist strictly increasing sequences of positive integers $\{\kappa_n\}$, $\{\nu_n\}$ and $\{\lambda_n\}$ such that if $x$ and $y$ are any two points on the unit circle then, for all integers $n \geq 0$,*

$$|Q_n(x) - Q_n(y)| \leq \kappa_n |x-y|; \tag{2.7}$$

$$|P_n(x) - P_n(y)| \leq \nu_n |x-y|; \tag{2.8}$$

*and*

$$|\chi_n(x) - \chi_n(y)| \leq \alpha_n |x-y|. \tag{2.9}$$

*Proof.* Let $\{f_n(q)\}$ be any sequence of polynomials in $\mathbb{Z}[q]$. Suppose $f_n(q) = \sum_{i=0}^{M_n} \gamma_i q^i$, where the $\gamma_i$'s are in $\mathbb{Z}$. Then

$$|f_n(x) - f_n(y)| \leq \sum_{i=1}^{M_n} |\gamma_i|\left|x^i - y^i\right| \leq \sum_{i=1}^{M_n} i\,|\gamma_i||x-y|.$$

Now set $\delta_n = \max\left\{ \sum_{i=1}^{M_n} i\,|\gamma_i|,\ 1,\ \delta_{n-1}+1\right\}$. Inequality (2.7) follows by setting $f_n(q) = Q_n(q)$ and $\delta_n = \kappa_n$. The result for (2.8) and (2.9) follow similarly. □

**Lemma 2.** *Let* $G(q)$ *be as in* (2.1) *and* $\chi_n(q)$*as in* (2.3)*. Then, for* $n \geq 1$*,*

$$P_n(q)Q_{n-1}(q) - P_{n-1}(q)Q_n(q) = (-1)^{n-1}\chi_n(q). \tag{2.10}$$

*Proof.* This follows from (2.2). □

Let $\{\kappa_n\}_{n=1}^{\infty}$ be as in (2.7), $\{\alpha_n\}_{n=1}^{\infty}$ be as in (2.9), and $\{n(m)\}_{m=1}^{\infty}$, $j$ and $d$ as in the statement of Theorem 3. Let $S_0$ be the set of all irrational $t \in (0,1)$ satisfying

$$e_1(t) \equiv j \pmod{d}, \tag{2.11}$$

and, for $i \geq 1$,

$$e_{2i+1}(t) \equiv 0 \pmod{d}, \tag{2.12}$$

and

$$e_{2i+2}(t) \geq \frac{2\pi}{d_{2i+1}^2} \max\left\{\kappa_{n(d_{2i+1})},\ \frac{2\alpha_{n(d_{2i+1})}}{C_1}\right\} \text{ infinitely often.} \tag{2.13}$$

Note that, for $i \geq 1$, $d_{2i+1}(t) \equiv j \pmod{d}$, so that $n(d_{2i+1})$ is well-defined. Note also that $S_0$ is an uncountable set.

**Lemma 3.** *For* $t \in S_0$*, we have*

$$\left| t - \frac{c_{2i+1}(t)}{d_{2i+1}(t)}\right| < \frac{1}{2\pi\kappa_{n(d_{2i+1})}} \tag{2.14}$$

*and*

$$\left| t - \frac{c_{2i+1}(t)}{d_{2i+1}(t)}\right| < \frac{C_1}{4\pi\alpha_{n(d_{2i+1})}} \tag{2.15}$$

*for infinitely many* $i$*.*

*Proof.* Let $i$ be one of the infinitely many integers for which

$$e_{2i+2}(t) \geq \frac{2\pi}{d_{2i+1}^2} \max\left\{\kappa_{n(d_{2i+1})},\ \frac{2\alpha_{n(d_{2i+1})}}{C_1}\right\}$$

and let $t_{2i+2} = [e_{2i+2}(t), e_{2i+3}(t), \cdots]$ denote the $2i+2$-th tail of the continued fraction expansion for $t$. Then

$$\begin{aligned}\left| t - \frac{c_{2i+1}(t)}{d_{2i+1}(t)}\right| &= \left| \frac{t_{2i+2}c_{2i+1}(t) + c_{2i}(t)}{t_{2i+2}d_{2i+1}(t) + d_{2i}(t)} - \frac{c_{2i+1}(t)}{d_{2i+1}(t)}\right| \\ &= \frac{1}{d_{2i+1}(t)(t_{2i+2}d_{2i+1}(t) + d_{2i}(t))} \\ &< \frac{1}{d_{2i+1}(t)^2 e_{2i+2}(t)} \leq \frac{1}{2\pi\kappa_{n(d_{2i+1})}}.\end{aligned}$$

(2.15) follows similarly. □

*Proof of Theorem 3:* Let $t \in S_0$, let $y = \exp(2\pi i t)$, and for $i \geq 1$, set $x_i = \exp(2\pi i c_{2i+1}(t)/d_{2i+1}(t)) = \exp(2\pi i c_{2i+1}/d_{2i+1})$, so that $x_i$ is a primitive $d_{2i+1}$-th root of unity, where $d_{2i+1} \equiv j \pmod{d}$. We use, in turn, (2.7), the fact that chord length is shorter than arc length, and (2.14), to get that, for infinitely many $i$,

$$\begin{aligned}|Q_{n(d_{2i+1})}(y) - Q_{n(d_{2i+1})}(x_i)| &\leq \kappa_{n(d_{2i+1})}|y - x_i| \\ &< 2\pi\kappa_{n(d_{2i+1})}\left|t - \frac{c_{2i+1}(t)}{d_{2i+1}(t)}\right| < 1.\end{aligned} \tag{2.16}$$

Similarly,

$$\begin{aligned}|Q_{(n(d_{2i+1})-1)}(y) - Q_{(n(d_{2i+1})-1)}(x_i)| &\leq \kappa_{(n(d_{2i+1})-1)}|y - x_i| \\ &< \frac{\kappa_{(n(d_{2i+1})-1)}}{\kappa_{n(d_{2i+1})}} < 1.\end{aligned} \tag{2.17}$$

The last inequality follows since the sequence $\{\kappa_n\}$ is strictly increasing. Apply the triangle inequality to (2.16) and (2.17) and use (2.5) to get that

$$|Q_{n(d_{2i+1})}(y)| < 1 + C_2 \tag{2.18}$$

and

$$|Q_{n(d_{2i+1})-1}(y)| < 1 + C_2 \tag{2.19}$$

Similarly, use (2.9), the fact that chord length is shorter than arc length and (2.15) to get that, for the same infinite set of $i$'s,

$$\begin{aligned}||\chi_{n(d_{2i+1})}(y)| - |\chi_{n(d_{2i+1})}(x_i)|| &\leq |\chi_{n(d_{2i+1})}(y) - \chi_{n(d_{2i+1})}(x_i)| \\ &\leq \alpha_{n(d_{2i+1})}|y - x_i| \\ &< 2\pi\alpha_{n(d_{2i+1})}\left|t - \frac{c_{2i+1}(t)}{d_{2i+1}(t)}\right| \\ &< \frac{C_1}{2}.\end{aligned} \tag{2.20}$$

Use the reverse triangle inequality and (2.4)to get that

$$|\chi_{n(d_{2i+1})}(y)| > |\chi_{n(d_{2i+1})}(x_i)| - \frac{C_1}{2} \geq \frac{C_1}{2}. \tag{2.21}$$

Use (2.18), (2.19) and (2.21) to get that

$$\begin{aligned}\left|\frac{P_{n(d_{2i+1})}(y)}{Q_{n(d_{2i+1})}(y)} - \frac{P_{n(d_{2i+1})-1}(y)}{Q_{n(d_{2i+1})-1}(y)}\right| &= \frac{|\chi_{n(d_{2i+1})}(y)|}{|Q_{n(d_{2i+1})}(y)Q_{n(d_{2i+1})-1}(y)|} \\ &> \frac{C_1}{2(1+C_2)^2}.\end{aligned}$$

Since this holds for the infinite set of integers $\{n(d_{2i+1})\}_{i=1}^{\infty}$, it follows that $G(y)$ does not converge to a finite value. Since $S_0$ is an uncountable set, this proves the theorem.

□

Before proving a corollary to this theorem we need the following proposition.

**Proposition 1.** *Let $q$ be a primitive $2t+1$-th root of unity. Then*

$$\prod_{i=1}^{2t}(1+q^i)=1. \tag{2.22}$$

*Proof.* Suppose $q=\exp(2\pi ia/(2t+1))$, for some $a\in\mathbb{N}$, where $(a,2t+1)=1$. Then

$$\prod_{i=1}^{2t}(x-q^i)=\frac{x^{2t+1}-1}{x-1}.$$

Let $x=-1$ to get the result. □

We now prove the following corollary to Theorem 3.

**Corollary 2.** *For each of the continued fractions $K(q)$, $S_1(q)$, $S_2(q)$ and $S_3(q)$, there exists an uncountable set of points on the unit circle at which the continued fraction does not converge to a finite value.*

*Proof.* We use some of the information contained in Table 1.

| $G(q)$ | $K(q)$ | $S_1(q)$ | $S_2(q)$ | $S_3(q)$ |
|---|---|---|---|---|
| $(j,d)$ | $(1,5)$ | $(1,4)$ | $(1,8)$ | $(1,6)$ |
| $n(m)$ | $m-1$ | $2m-1$ | $2m-1$ | $m-1$ |
| $Q_{n(m)-1}$ | 0 | 1 | 1 | 0 |
| $Q_{n(m)}$ | $q^{(m-1)/5}$ | $(-1)^{(m-1)/4}\,q^{(m^2-1)/8}$ | $q^{(m-1)/2}$ | $q^{(m-1)/3}$ |

TABLE 1

In each case, $q$ is a primitive $m$-th root of unity and $m\equiv j \mod d$, where $(j,d)$ is the pair of integers in the statement of Theorem 3. The values in the table come from the papers of Schur [8] and Zhang [9].

For $K(q)$, take $(j,d)=(1,5)$ and for $m\equiv 1 \pmod 5$, set $n(m)=m-1$. From (1.1) we can take $C_1=1$ and from Table 1 we can also take $C_2=1$. This proves the result for $K(q)$.

For $S_1(q)$ take $(j,d)=(1,4)$ and for $m\equiv 1 \pmod 4$, $m=2t+1$, for some positive integer $t$, set $n(m)=2m-1$. Let $x_m$ be a primitive $m$-th root of unity. Then, by (2.22) and (1.4),

$$|\chi_{n(m)}(x_m)|=\prod_{i=1}^{2t}|(1+x_m^i)|=1.$$

Once again (2.4) is satisfied with $C_1=1$. From Table 1 we can once again take $C_2=1$. This proves the result for $S_1(q)$.

For $S_2(q)$ take $(j,d)=(1,8)$ and for $m\equiv 1 \pmod 8$, $m=2t+1$, for some positive integer $t$, let $n(m)=2m-1$. Let $x_m$ be a primitive $m$-th root of unity. Then, from (1.5),

$$\begin{aligned}
\left|\chi_{n(m)}(x_m)\right| &= \prod_{i=1}^{2t+1}\left|(1+x_m^{2i-1})\right| \\
&= \prod_{i=1}^{t}\left|(1+x_m^{2i-1})\right|\left|(1+x^{2t+1})\right|\prod_{i=t+2}^{2t+1}\left|(1+x_m^{2i-1})\right| \\
&= 2\prod_{i=1}^{t}\left|(1+x_m^{2i-1})\right|\prod_{i=1}^{t}\left|(1+x_m^{2i})\right| \\
&= 2\prod_{i=1}^{2t}\left|(1+x_m^{i})\right| = 2.
\end{aligned}$$

The last equality uses (2.22). In this case, (2.4) is satisfied with $C_1=2$. From Table 1 we can once again take $C_2=1$. This proves the result for $S_2(q)$.

Finally, for $S_3(q)$ take $(j,d)=(1,6)$ and for $m\equiv 1 \pmod 6$, $m=2t+1$, for some positive integer $t$, let $n(m)=m-1$. Let $x_m$ be a primitive $m$-th root of unity. By (2.22),

$$|\chi_{n(m)}(x_m)| = \prod_{i=1}^{2t}|(1+x_m^{i})| = 1.$$

Once again (2.4) is satisfied with $C_1=1$. From Table 1, it is clear that we can take $C_2=1$. This proves the result for $S_3(q)$. □

## 3. Concluding Remarks

In proving the existence of an uncountable set of points on the unit circle at which a $q$-continued fraction $G(q)$ does not converge to finite values, our methods rely on knowing the behavior of the continued fraction at roots of unity and, if $q$ is a primitive$m$-th root of unity, on knowing something about the values of$|\chi_{n(m)}(q)|$, $|Q_{n(m)}(q)|$ and $|Q_{n(m)-1}(q)|$ (see the statement of Theorem 3). It would be interesting to have a criterion based on the $a_n(q)$ and the $b_n(q)$ which would indicate whether (2.4) and (2.5)were satisfied, as this would automatically give information about the convergence behavior of the continued fraction on the unit circle away from roots of unity.

Our Theorem 3 can show the existence of a set of measure zero on the unit circle at which a $q$-continued fraction does not converge to finite values. However, it seems likely that the particular continued fractions that we have looked at diverge almost everywhere, in both the classical and general senses, on the unit circle. As yet, we do not see how to prove this almost everywhere divergence.

The most famous $q$-continued fraction after the Rogers-Ramanujan continued fraction is the Göllnitz-Gordon continued fraction,

$$GG(q) := 1 + q + \frac{q^2}{1+q^3} + \frac{q^4}{1+q^5} + \frac{q^6}{1+q^7} + \cdots. \tag{3.1}$$

This continued fraction tends to the same limit as $S_2(q)$, for each $q$ inside the unit circle, but the behavior at roots of unity is slightly different. Based on computer investigations of the behavior of $GG(q)$ at roots of unity, it would seem that the following is true. Both $GG(q)$ and $S_2(q)$ agree at $m$-th roots of unity (either both converge to the same limit or both diverge), except if $m \equiv 2 (\text{mod } 4)$, in which case $GG(q)$ diverges and $S_2(q)$ converges. Also, computer evidence also seems to suggest that $GG(q)$satisfies the conditions on the $a_i(q)$ and the $Q_i(q)$ required by Theorem 3, implying that there is an uncountable set of points on the unit circle at which the Göllnitz-Gordon continued fraction does not converge to finite values. However, these facts have not yet been proved. We hope to do this in a later paper and thereby show that the Göllnitz-Gordon continued fraction does indeed diverge at uncountably many points on the unit circle.

In [3] we extend our results in [2] on the divergence in the *general* sense (see, for example, [5] and [6]) of the Rogers-Ramanujan continued fraction on the unit circle. We show, for each $q$-continued fraction in a certain class of $q$-continued fractions (a class which includes the Rogers-Ramanujan continued fraction and the three Ramanujan-Selberg continued fractions), that there exists an uncountable set of points on the unit circle at which the continued fraction diverges in the general sense.

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