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A THEOREM ON DIVERGENCE IN THE GENERAL SENSE FOR CONTINUED FRACTIONS

DOUGLAS BOWMAN AND JAMES MC LAUGHLIN

Abstract. If the odd and even parts of a continued fraction converge to different values, the continued fraction may or may not converge in the general sense. We prove a theorem which settles the question of general convergence for a wide class of such continued fractions.

We apply this theorem to two general classes of q continued fraction to show, that if $G(q)$ is one of these continued fractions and $|q| > 1$, then either $G(q)$ converges or does not converge in the general sense.

We also show that if the odd and even parts of the continued fraction $K_{n=1}^{\infty} a_n/1$ converge to different values, then $\lim_{n\to\infty} |a_n| = \infty$.

1. Introduction

In [7], Jacobsen revolutionised the subject of the convergence of continued fractions by introducing the concept of *general convergence*. General convergence is defined in [9] as follows.

Let the n-th approximant of the continued fraction

$$M = b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{b_3 + \dots}}} \tag{1.1}$$

be denoted by A_n/B_n (A_n is the n-th *numerator* convergent and B_n is the n-th *denominator* convergent) and let

$$S_n(w) = \frac{A_n + wA_{n-1}}{B_n + wB_{n-1}}.$$

Define the chordal metric d on $\hat{\mathbb{C}}$ by

$$d(w,z) = \frac{|z-w|}{\sqrt{1+|w|^2}\sqrt{1+|z|^2}}$$

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when w and z are both finite, and

$$d(w,\infty)=\frac{1}{\sqrt{1+|w|^2}}.$$

Definition: The continued fraction M is said to *converge generally* to $f\in\hat{\mathbb{C}}$ if there exist sequences $\{v_n\}$, $\{w_n\}\subset\hat{\mathbb{C}}$ such that $\liminf d(v_n,w_n)>0$ and

$$\lim_{n\to\infty}S_n(v_n)=\lim_{n\to\infty}S_n(w_n)=f.$$

Remark: Jacobson shows in [7] that, if a continued fraction converges in the general sense, then the limit is unique.

The idea of general convergence is of great significance because classical convergence implies general convergence (take $v_n=0$ and $w_n=\infty$, for all n), but the converse does not necessarily hold. General convergence is a natural extension of the concept of classical convergence for continued fractions.

The *even* part of the continued fraction M at (1.1) is the continued fraction whose n-th numerator (denominator) convergent equals A_{2n} (B_{2n}), for $n\geq 0$. The *odd* part of M is the continued fraction whose zero-th numerator convergent is A_1/B_1, whose zero-th denominator convergent is 1, and whose n-th numerator (respectively denominator) convergent equals A_{2n+1} (respectively B_{2n+1}), for $n\geq 1$.

In this present paper we investigate the general convergence of continued fractions whose odd and even parts each converge, but to different values. Such continued fractions may or may not converge in the general sense as the following examples show.

Example 1. *Let*

$$K(q)=1+\frac{q}{1}\,+\,\frac{q^2}{1}\,+\,\frac{q^3}{1}\,+\cdots+\,\frac{q^n}{1}\,+\cdots. \tag{1.2}$$

If $|q|>1$ then the odd and even parts of $K(q)$ converge but $K(q)$ does not converge generally.

The continued fraction $K(q)$ is the famous Rogers-Ramanujan continued fraction. It was stated without proof by Ramanujan that, if $|q|>1$, then the odd part of $K(q)$ converges to $1/K(-1/q)$ and the even part converges to $q\,K(1/q^4)$ (See Entry 59 of [1] for a proof of Ramanujan's claim). However, $K(q)$ is easily seen to be equivalent to the following continued fraction:

$$\hat{K}(q):=1+\frac{1}{1/q}\,+\,\frac{1}{1/q}\,+\,\frac{1}{1/q^2}\,+\,\frac{1}{1/q^2}\,+\cdots+\,\frac{1}{1/q^n}\,+\,\frac{1}{1/q^n}\,+\cdots.$$

It is an easy consequence of the Stern-Stolz Theorem below, as extended by Lorentzen and Waadeland, that this continued fraction does not converge in the general sense for any q outside the unit circle.

Example 2. *Let*

$$G:=\frac{2}{1}\,+\,\frac{-1}{2}\,+\,K_{n=3}^{\infty}\frac{a_n}{b_n},$$

where

$$a_{2n+1} = 1 + \frac{1}{2\,n^2} + \frac{1}{n}, \qquad b_{2n+1} = \frac{-1}{2\,n^3},$$
$$a_{2n+2} = \frac{2\,(1+n)^3}{n\,\left(1+2\,n+2\,n^2\right)}, \qquad b_{2n+2} = \frac{1+n}{1+2\,n+2\,n^2}.$$

Then the odd and even parts of G *tend to different values and* G *converges in the general sense.*

Proof. It is easy to check that the numerators A_n and denominators B_n satisfy

$$A_{2n-1} = n+1, \qquad A_{2n} = n + 3n^2,$$
$$B_{2n-1} = n, \qquad B_{2n} = n^2.$$

Thus the odd approximants tend to 1 and the even approximants tend to 3. Observe that

$$\frac{A_{2n} + w_{2n}A_{2n-1}}{B_{2n} + w_{2n}B_{2n-1}} = \frac{n + 3\,n^2 + (1+n)\; w_{2n}}{n^2 + n\,w_{2n}},$$

$$\frac{A_{2n+1} + w_{2n+1}A_{2n}}{B_{2n+1} + w_{2n+1}B_{2n}} = \frac{2 + n + \left(n + 3\,n^2\right)\; w_{2n+1}}{1 + n + n^2\,w_{2n+1}}.$$

Each of these expressions converges to 3, when, for example, $\{w_n\}$ is the constant sequence with value 1 and when it is the constant sequence with value 2. Thus the continued fraction converges generally to 3. □

It is therefore desirable to have criteria, based on the partial quotients of a continued fraction, for determining whether a continued fraction whose odd and even parts converge diverges in the general sense.

An example of a theorem on divergence in the general sense is the Stern-Stolz Theorem, as extended by Lorentzen and Waadeland.

Theorem 1. (***The Stern-Stolz Theorem*** ([9], *p.94*)) *The continued fraction* $b_0 + K_{n=1}^{\infty} 1/b_n$ *diverges generally if* $\sum |b_n| < \infty$. *In fact,*

$$\lim_{n\to\infty} A_{2n+p} = P_p \neq \infty, \qquad \lim_{n\to\infty} B_{2n+p} = Q_p \neq \infty,$$

for $p = 0, 1$, *where*

$$P_1 Q_0 - P_0 Q_1 = 1.$$

However, if a continued fraction is not already of the form $K_{n=1}^{\infty} 1/b_n$, these b_n may become quite complicated once an equivalence transformation is applied to the continued fraction to bring it to this form and it may not be so easy to determine if the series $\sum |b_n|$ converges.

In the present paper, we prove a theorem which gives a simple criterion, based on the partial quotients, for deciding if a continued fraction diverges in the general sense, provided it is known that the odd- and even parts converge and whether these limits are equal.

We then apply our theorem to two classes of q-continued fraction described in our paper [6] to show that if $|q_1| > 1$ and $H(q)$ is a continued fraction in either class, then either $H(q_1)$ converges or does not converge generally.

2. A Theorem on Divergence in the General Sense

We now prove the following theorem.

Theorem 2. *Let the odd and even parts of the continued fraction* $C = b_0 + K_{n=1}^{\infty} a_n/b_n$ *converge to different limits. Further suppose that there exist positive constants* c_1, c_2 *and* c_3 *such that, for* $i \geq 1$,

$$c_1 \leq |b_i| \leq c_2 \tag{2.1}$$

and

$$\left| \frac{a_{2i+1}}{a_{2i}} \right| \leq c_3. \tag{2.2}$$

Then C *does not converge generally.*

Remark: It might seem that Condition 2.1 prevents the application of this theorem to continued fractions $K_{n=1}^{\infty} a_n/b_n$ in which the b_n become unbounded but a similarity transformation to put the continued fraction in the form $b_0 + K_{n=1}^{\infty} c_n/1$ removes this difficulty.

Proof of Theorem 2. Let the i-th approximant of $C = b_0 + K_{n=1}^{\infty} a_n/b_n$ be denoted by A_i/B_i. Suppose the odd approximants tend to f_1 and that the even approximants tend to f_2. Further suppose that C converges generally to $f \in \hat{\mathbb{C}}$ and that $\{v_n\}$, $\{w_n\} \subset \hat{\mathbb{C}}$ are two sequences such that

$$\lim_{n\to\infty} \frac{A_n + v_n A_{n-1}}{B_n + v_n B_{n-1}} = \lim_{n\to\infty} \frac{A_n + w_n A_{n-1}}{B_n + w_n B_{n-1}} = f$$

and

$$\liminf_{n\to\infty} d(v_n, w_n) > 0.$$

It will be shown that these two conditions lead to a contradiction. Suppose first that $|f| < \infty$ and, without loss of generality, that $f \neq f_1$. (If $f = f_1$, then $f \neq f_2$ and we proceed similarly). We write

$$\frac{A_n + w_n A_{n-1}}{B_n + w_n B_{n-1}} = f + \gamma_n, \qquad \frac{A_n + v_n A_{n-1}}{B_n + v_n B_{n-1}} = f + \gamma_n^{'},$$

where $\gamma_n \to 0$ and $\gamma_n^{'} \to 0$ as $n \to \infty$. By assumption it follows that $A_{2n} = B_{2n}(f_2 + \alpha_{2n})$ and $A_{2n+1} = B_{2n+1}(f_1 + \alpha_{2n+1})$, where $\alpha_i \to 0$ as $i \to \infty$. Then

$$\begin{aligned}\frac{A_{2n} + w_{2n} A_{2n-1}}{B_{2n} + w_{2n} B_{2n-1}} &= \frac{B_{2n}(f_2 + \alpha_{2n}) + w_{2n} B_{2n-1}(f_1 + \alpha_{2n-1})}{B_{2n} + w_{2n} B_{2n-1}} \\ &= f + \gamma_{2n}.\end{aligned}$$

By simple algebra we have

$$w_{2n} = \frac{B_{2\,n}\,(-f + f_2 + \alpha_{2\,n} - \gamma_{2\,n})}{B_{2\,n-1}\,(f - f_1 - \alpha_{2\,n-1} + \gamma_{2\,n})}.$$

Similarly,

$$v_{2n} = \frac{B_{2\,n}\,\left(-f + f_2 + \alpha_{2\,n} - \gamma^{'}_{2\,n}\right)}{B_{2\,n-1}\,\left(f - f_1 - \alpha_{2\,n-1} + \gamma^{'}_{2\,n}\right)}.$$

Note that $B_{2\,n}$, $B_{2\,n-1} \neq 0$ for n sufficiently large, since the odd and even parts of the continued fraction converge. If $f \neq f_2$, then

$$\lim_{n\to\infty} d(v_{2n}, w_{2n}) \leq \lim_{n\to\infty} \frac{|v_{2n} - w_{2n}|}{|w_{2n}|} = 0.$$

Hence $f = f_2$,

$$w_{2n} = \frac{B_{2\,n}\,(\alpha_{2\,n} - \gamma_{2\,n})}{B_{2\,n-1}\,(f - f_1 - \alpha_{2\,n-1} + \gamma_{2\,n})}$$

and

$$v_{2n} = \frac{B_{2\,n}\,(\alpha_{2\,n} - \gamma^{'}_{2\,n})}{B_{2\,n-1}\,\left(f - f_1 - \alpha_{2\,n-1} + \gamma^{'}_{2\,n}\right)}.$$

Now we show that

$$\lim_{n\to\infty} \left|\frac{B_{2n}}{B_{2n-1}}\right| = \infty.$$

For if not, then there is a sequence $\{n_i\}$ and a positive constant M such that $|B_{2n_i}/B_{2n_i-1}| \leq M$ for all n_i, and then

$$\begin{aligned}
&\lim_{i\to\infty} d(v_{2n_i}, w_{2n_i}) \leq \lim_{i\to\infty} |v_{2n_i} - w_{2n_i}| \\
&\leq \lim_{i\to\infty} M \left| \frac{\alpha_{2\,n_i} - \gamma^{'}_{2\,n_i}}{f - f_1 - \alpha_{2\,n_i-1} + \gamma^{'}_{2\,n_i}} - \frac{\alpha_{2\,n_i} - \gamma_{2\,n_i}}{f - f_1 - \alpha_{2\,n_i-1} + \gamma_{2\,n_i}} \right| = 0.
\end{aligned}$$

Similarly, after substituting f_2 for f, we have that

$$w_{2n+1} = \frac{B_{2n+1}}{B_{2n}} \left(\frac{f_1 - f_2 + \alpha_{2n+1} - \gamma_{2n+1}}{\gamma_{2n+1} - \alpha_{2n}} \right)$$

and

$$v_{2n+1} = \frac{B_{2n+1}}{B_{2n}} \left(\frac{f_1 - f_2 + \alpha_{2n+1} - \gamma^{'}_{2n+1}}{\gamma^{'}_{2n+1} - \alpha_{2n}} \right).$$

We now show that

$$\lim_{n\to\infty} \left|\frac{B_{2n+1}}{B_{2n}}\right| = 0.$$

If not, then there is a sequence $\{n_i\}$ and some $M > 0$ such that $|B_{2n_i+1}/B_{2n_i}| \geq M$ for all n_i. Then $\lim_{i\to\infty} w_{2n_i+1} = \lim_{i\to\infty} v_{2n_i+1} = \infty$ and $\lim_{i\to\infty} d(v_{2n_i+1}, w_{2n_i+1}) = 0$.

Finally, we show that it is impossible to have both $\lim_{n\to\infty}|B_{2n+1}/B_{2n}| = 0$ and $\lim_{n\to\infty}|B_{2n}/B_{2n-1}| = \infty$. For ease of notation let B_n/B_{n-1} be denoted by r_n, so that $r_{2n} \to \infty$ and $r_{2n+1} \to 0$, as $n \to \infty$. From the recurrence relations for the B_i's, namely, $B_i = b_i B_{i-1} + a_i B_{i-2}$, we have

$$r_{2n}(r_{2n+1} - b_{2n+1}) = a_{2n+1}$$

and

$$r_{2n-1}(r_{2n} - b_{2n}) = a_{2n}.$$

Thus

$$\frac{r_{2n}}{r_{2n} - b_{2n}} = \frac{a_{2n+1} r_{2n-1}}{a_{2n}(r_{2n+1} - b_{2n+1})},$$

and by (2.1) and (2.2) the left side tends to 1 and the right side tends to 0, as $n \to \infty$, giving the required contradiction.

If $f = \infty$, then we write

$$\frac{A_n + w_n A_{n-1}}{B_n + w_n B_{n-1}} = \frac{1}{\gamma_n},$$

$$\frac{A_n + v_n A_{n-1}}{B_n + v_n B_{n-1}} = \frac{1}{\gamma'_n},$$

where $\lim_{n\to\infty}\gamma_n = \lim_{n\to\infty}\gamma'_n = 0$. With the α_i's as above we find that

$$w_{2n} = -\frac{B_{2\,n}\,(-1 + f_2\,\gamma_{2\,n} + \alpha_{2\,n}\,\gamma_{2\,n})}{B_{2\,n-1}\,(-1 + f_1\,\gamma_{2\,n} + \alpha_{2\,n+1}\,\gamma_{2\,n})}$$

and

$$v_{2n} = -\frac{B_{2\,n}\left(-1 + f_2\,\gamma'_{2\,n} + \alpha_{2\,n}\,\gamma'_{2\,n}\right)}{B_{2\,n-1}\left(-1 + f_1\,\gamma'_{2\,n} + \alpha_{2\,n+1}\,\gamma'_{2\,n}\right)}.$$

In this case it follows easily that $\lim_{n\to\infty} d(w_{2n}, v_{2n}) = 0$.

□

3. Application to q-Continued Fractions

In [6], one type of continued fraction we considered was of the form

$$G(q) := 1 + K_{n=1}^{\infty}\frac{a_n(q)}{1} := 1 + \frac{f_1(q^0)}{1} + \cdots + \frac{f_k(q^0)}{1} + \frac{f_1(q^1)}{1} + \cdots + \frac{f_k(q^1)}{1} + \cdots + \frac{f_1(q^n)}{1} + \cdots + \frac{f_k(q^n)}{1} + \cdots,$$

where $f_s(x) \in \mathbb{Z}[q][x]$, for $1 \le s \le k$. Thus, for $n \ge 0$ and $1 \le s \le k$,

$$a_{nk+s}(q) = f_s(q^n). \tag{3.1}$$

Many well-known q-continued fractions, including the Rogers-Ramanujan continued fraction at (1.2) and the three Ramanujan-Selberg continued fractions studied by Zhang in [10], namely,

$$S_1(q) := 1 + \frac{q}{1} + \frac{q+q^2}{1} + \frac{q^3}{1} + \frac{q^2+q^4}{1} + \cdots,$$

$$S_2(q) := 1 + \frac{q+q^2}{1} + \frac{q^4}{1} + \frac{q^3+q^6}{1} + \frac{q^8}{1} + \cdots,$$

and

$$S_3(q) := 1 + \frac{q+q^2}{1} + \frac{q^2+q^4}{1} + \frac{q^3+q^6}{1} + \frac{q^4+q^8}{1} + \cdots,$$

are of this form, with k at most 2. Following the example of these four continued fractions, we made the additional assumptions that, for $i \geq 1$,

$$\text{degree}(a_{i+1}(q)) = \text{degree}(a_i(q)) + C_3, \tag{3.2}$$

where C_3 is a fixed positive integer, and that all of the polynomials $a_n(q)$ had the same leading coefficient. The odd- and even parts of each of the four continued fractions above converge for $|q| > 1$, (see [1] and [10], where the authors also determined the limits). In [6], we extended these results on convergence outside the unit circle to the class of continued fractions described above. We proved the following theorem [6]:

Theorem 3. [6] *Suppose $G(q) = 1 + K_{n=1}^{\infty} a_n(q)/1$ is such that the $a_n := a_n(q)$ satisfy* (3.1) *and* (3.2). *Suppose further that each $a_n(q)$ has the same leading coefficient. If $|q| > 1$ then the odd and even parts of $G(q)$ both converge.*

It is now an easy matter to apply our Theorem 2 to the continued fractions of Theorem 3 to conclude that for each q outside the unit circle, either the continued fraction converges or does not converge generally. As an illustration we have the following example.

Example 3. *Let*

$$G_1(q) = 1 + \frac{6q}{1} + \frac{3q^2+7q}{1} + \frac{3q^3+5q^2}{1} + \frac{q^4+7q^3+3q+2}{1} + \frac{q^5+3q^4+2q^3}{1} + \frac{q^6+2q^5+7q^3}{1} + \frac{q^7+7q^5}{1} + \frac{q^8+7q^6+3q^3+2q}{1} + \cdots$$
$$\cdots + \frac{q^{4n+1}+3q^{3n+1}+2q^{2n+1}}{1} + \frac{q^{4n+2}+2q^{3n+2}+7q^{2n+1}}{1} + \frac{q^{4n+3}+5q^{3n+2}+2q^{2n+3}}{1} + \frac{q^{4n+4}+7q^{3n+3}+3q^{2n+1}+2q^n}{1} + \cdots.$$

If $|q| > 1$, then the odd and even parts of $G_1(q)$ converge. If the odd and even parts are not equal, then $G_1(q)$ does not converge generally.

In [6] we also studied continued fractions of the form

$$\begin{aligned} G(q) &:= b_0(q)+K_{n=1}^{\infty}\frac{a_n(q)}{b_n(q)} \\ &:= g_0(q^0)+\frac{f_1(q^0)}{g_1(q^0)} + \cdots + \frac{f_{k-1}(q^0)}{g_{k-1}(q^0)} + \frac{f_k(q^0)}{g_0(q^1)} \\ &\quad + \frac{f_1(q^1)}{g_1(q^1)} + \cdots + \frac{f_{k-1}(q^1)}{g_{k-1}(q^1)} + \frac{f_k(q^1)}{g_0(q^2)} + \\ \cdots + \frac{f_k(q^{n-1})}{g_0(q^n)} &+ \frac{f_1(q^n)}{g_1(q^n)} + \cdots + \frac{f_{k-1}(q^n)}{g_{k-1}(q^n)} + \frac{f_k(q^n)}{g_0(q^{n+1})} + \cdots \end{aligned}$$

where $f_s(x), g_{s-1}(x) \in \mathbb{Z}[q][x]$, for $1 \leq s \leq k$. Thus, for $n \geq 0$ and $1 \leq s \leq k$,

$$a_{nk+s}(q) = f_s(q^n), \qquad b_{nk+s-1}(q) = g_{s-1}(q^n). \tag{3.3}$$

An example of a continued fraction of this type is the Göllnitz-Gordon continued fraction $(k = 1)$,

$$GG(q) := 1 + q + \frac{q^2}{1+q^3} + \frac{q^4}{1+q^5} + \frac{q^6}{1+q^7} + \cdots.$$

We restricted the type of continued fraction examined as follows. We supposed that degree $(a_1(q)) = r_1$, degree $(b_0(q)) = r_2$, and that, for $i \geq 1$,

$$\begin{aligned} \text{degree}(a_{i+1}(q)) &= \text{degree}(a_i(q)) + a, \\ \text{degree}(b_i(q)) &= \text{degree}(b_{i-1}(q)) + b, \end{aligned} \tag{3.4}$$

where a and b are fixed positive integers and r_1 and r_2 are non-negative integers. Condition 3.4 means that, for $n \geq 1$, $a_n(q)$ has degree equal to $(n-1)a + r_1$ and that $b_n(q)$ has degree equal to $n\,b + r_2$. We also supposed that each $a_n(q)$ has the same leading coefficient L_a and that each $b_n(q)$ has the same leading coefficient L_b.

For such continued fractions we had the following theorem [6]:

Theorem 4. [6] *Suppose* $G(q) = b_o + K_{n=1}^{\infty} a_n(q)/b_n(q)$ *is such that the* $a_n := a_n(q)$ *and the* $b_n := b_n(q)$ *satisfy* (3.3) *and* (3.4)*. Suppose further that each* $a_n(q)$ *has the same leading coefficient* L_a *and that each* $b_n(q)$ *has the same leading coefficient* L_b*. If* $2b > a$ *then* $G(q)$ *converges everywhere outside the unit circle. If* $2b = a$*, then* $G(q)$ *converges outside the unit circle to values in* $\hat{\mathbb{C}}$*, except possibly at points* q *satisfying* $q^{b-r_1+2r_2} \in \big[-4\,L_a/L_b^2, 0\big)$ *or* $\big(0, -4\,L_a/L_b^2\big]$*, depending on the sign of* L_a*. If* $2b < a$*, then the odd and even parts of* $G(q)$ *converge everywhere outside the unit circle.*

Remark: Both our Theorem 3 and 4 were derived from theorems on limit–periodic continued fractions and give stronger results than can be derived from applying simple convergence criteria such as Worpitzky's Theorem.

Once again it is easy to apply our Theorem 2 to the continued fractions of Theorem 4 to conclude, in the case $2b < a$, that for each q outside the unit circle, either the continued fraction converges or does not converge generally. As an illustration we have the following example.

Example 4. *Let*

$$
\begin{aligned}
&G_2(q) := q+2+\\
&\frac{q^3+5q^2}{q^2+2}\;+\;\frac{q^6+2q^4+7q^2}{q^3+2}\;+\;\frac{q^9+2q^6+5q^4}{q^4+2}\;+\;\frac{q^{12}+7q^6+3q^2+2}{q^5+q+1}\;+\\
&\frac{q^{15}+3q^8+2q^6}{q^6+q^2+1}\;+\;\frac{q^{18}+2q^{10}+7q^6}{q^7+q^2+1}\;+\;\frac{q^{21}+7q^{10}}{q^8+q^3}\;+\;\frac{q^{24}+7q^{12}+3q^6+2q^2}{q^9+q^2+1}\\
&\quad+\cdots+\frac{q^{12n+3}+3q^{6n+2}+2q^{4n+2}}{q^{4n+2}+q^{2n}+1}\;+\;\frac{q^{12n+6}+2q^{6n+4}+7q^{4n+2}}{q^{4n+3}+q^{2n}+1}\\
&+\frac{q^{12n+9}+5q^{6n+4}+2q^{4n+6}}{q^{4n+4}+q^{3n}+1}\;+\;\frac{q^{12n+12}+7q^{6n+6}+3q^{4n+2}+2q^{2n}}{q^{4(n+1)+1}+q^{n+1}+1}\;+\;\cdots.
\end{aligned}
$$

If $|q|>1$, then the odd and even parts of $G_2(q)$ converge. If the odd and even parts are not equal, then $G_2(q)$ does not converge generally.

4. Continued fractions whose odd and even parts tend to different limits

Since our Theorem 2 deals with continued fractions whose odd and even parts converge to different values, it is desirable to know something about the form of such continued fractions. We have the following theorem.

Theorem 5. *Suppose the odd and even parts of the continued fraction $K_{n=1}^{\infty} a_n/1$ converge to different values. Then $\lim_{n\to\infty}|a_n| = \infty$.*

We need two preliminary results.

Lemma 1. *Suppose $\{K_n\}_{n=1}^{\infty}$ is the sequence of classical approximants of the continued fraction $K_{n=1}^{\infty} a_n/1$, where $a_n \neq 0$, for $n \geq 1$. If the continued fraction $K_{n=1}^{\infty} c_n/1$ also has $\{K_n\}_{n=1}^{\infty}$ as it sequence of classical approximants and $c_n \neq 0$, for $n \geq 1$, then $a_n = c_n$ for $n \geq 1$.*

Proof. Elementary. □

We also use the following result, proved by Daniel Bernoulli in 1775 [2] (see, for example, [8], pp. 11–12).

Proposition 1. *Let $\{K_0, K_1, K_2, \ldots\}$ be a sequence of complex numbers such that $K_i \neq K_{i-1}$, for $i = 1, 2, \ldots$. Then $\{K_0, K_1, K_2, \ldots\}$ is the sequence of approximants of the continued fraction*

$$
\begin{aligned}
&K_0 + \frac{K_1-K_0}{1} + \frac{K_1-K_2}{K_2-K_0} + \frac{(K_1-K_0)(K_2-K_3)}{K_3-K_1} +\\
&\qquad\cdots + \frac{(K_{n-2}-K_{n-3})(K_{n-1}-K_n)}{K_n-K_{n-2}} + \cdots\\
&\sim K_0 + \frac{K_1-K_0}{1} + \frac{\frac{K_1-K_2}{K_2-K_0}}{1} + \frac{\frac{(K_1-K_0)(K_2-K_3)}{(K_2-K_0)(K_3-K_1)}}{1} +
\end{aligned}
$$

$$\cdots + \frac{\frac{(K_{n-2}-K_{n-3})(K_{n-1}-K_n)}{(K_{n-1}-K_{n-3})(K_n-K_{n-2})}}{1} + \cdots .$$

Proof of Theorem 5. Let $\{K_n\}_{n=1}^{\infty}$ denote the sequence of classical approximants of the continued fraction $K_{n=1}^{\infty} a_n/1$. By assumption there exist $\alpha \neq \beta \in \mathbb{C}$ such that

$$\lim_{n\to\infty} K_{2n} = \alpha, \qquad \lim_{n\to\infty} K_{2n+1} = \beta.$$

Hence there exist two null sequences $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ such that

$$K_{2n} = \alpha + \alpha_n, \qquad K_{2n+1} = \beta + \beta_n. \tag{4.1}$$

By Lemma 1, Proposition 1 and (4.1), it follows that

$$\begin{aligned} a_{2n} &= \frac{(K_{2n-2} - K_{2n-3})(K_{2n-1} - K_{2n})}{(K_{2n-1} - K_{2n-3})(K_{2n} - K_{2n-2})} \\ &= \frac{(\alpha + \alpha_{n-1} - \beta - \beta_{n-2})(\beta + \beta_{n-1} - \alpha - \alpha_n)}{(\beta_{n-1} - \beta_{n-2})(\alpha_n - \alpha_{n-1})}. \end{aligned}$$

Since $\alpha \neq \beta$ and $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ are null sequences, it follows that $\lim_{n\to\infty} |a_{2n}| = \infty$. That $\lim_{n\to\infty} |a_{2n-1}| = \infty$ follows similarly.

□

5. Concluding Remarks

Let $m \geq 2$ be a positive integer. A continued fraction for which the odd- and even parts tend to different limits may be regarded as a special case ($m = 2$) of continued fractions for which the sequence of approximants in each arithmetic progression modulo m tends to a different limit. We will investigate such continued fractions in a later paper and also look at the question of whether or not they converge in the general sense.

We close with a question. Does there exist a continued fraction $K_{n=1}^{\infty} a_n/1$ whose odd and even parts converge to different values, for which the sequence $\{a_{2n+1}/a_{2n}\}$ is bounded and whose Stern–Stolz series diverges? This would mean that our Theorem 2 could show divergence in the general sense for a continued fraction that the Stern-Stolz Theorem could not be applied to.

On the other hand, it may be that if $\{a_n\}$ is any sequence of non-zero complex numbers such that the sequence $\{a_{2n+1}/a_{2n}\}$ is bounded and the continued fraction $K_{n=1}^{\infty} a_n/1$ is such that its odd and even parts converge to different values, then the Stern–Stolz series for $K_{n=1}^{\infty} a_n/1$ converges. A proof of this would be interesting. In this latter situation our Theorem 2 does not give anything new and may just be easier to apply to certain types of continued fraction.

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