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CONTROLLABILITY OF NEUTRAL STOCHASTIC INTEGRO-DIFFERENTIAL EVOLUTION EQUATIONS DRIVEN BY A FRACTIONAL BROWNIAN MOTION

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Abstract. We establish sufficient conditions for the controllability of a certain class of neutral stochastic functional integro-differential evolution equations in Hilbert spaces. The results are obtained using semigroup theory, resolvent operators and a fixed-point technique. An application to neutral integro-differential evolution equation perturbed by fractional Brownian motion is given.

Keywords: Neutral stochastic partial integro-differential equations, mild solutions, resolvent operators, fractional Brownian motion.

AMS Subject Classification: 35R10; 60G22; 60H20.

1. Introduction

The noise or perturbations of a system are typically modeled by a Brownian motion as such a process is Gauss-Markov and has independent increments. However, empirical data from many physical phenomena suggest that Brownian motion is often shown not to be an effective process to use in a model. A family of processes that seems to have wide physical applicability is fractional Brownian motion (fBm). This process was introduced by Kolmogorov in [10] and later studied by Mandelbrot and Van Ness in [11], where a stochastic integral representation in terms of a standard Brownian motion was obtained. Since the fBm B^H is not a semimartingale if $H \neq \frac{1}{2}$ (see [1]), we cannot use the classical Itô theory to construct a stochastic calculus with respect to fBm.

The subject of stochastic calculus with respect to fractional Brownian motion has gained considerable popularity and importance due to its frequent appearance in a wide variety of physical phenomena, such as hydrology, economic, telecommunications and medicine. Many contributions for stochastic calculus with respect to fBm have emerged in the last decades, see [3, 4, 6, 14]. For example, Ferrante and Rovira studied in [7] the existence and convergence when the delay goes to

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zero using the Riemann-Stieltjes integral. Using also the Riemann-Stieltjes integral, Boufoussi et al. [3] proved the existence and uniqueness of a mild solution and studied the dependence of the solution on the initial condition in infinite dimensional space. Recently, Caraballo et al. [5] and Boufoussi and Hajji [2] have discussed the existence, uniqueness and exponential asymptotic behavior of mild solutions using the Wiener integral. The literature related to neutral stochastic partial functional integro-differential equations driven by a fBm is not vast. Very recently, in [6], the authors studied the existence and uniqueness of mild solutions for a class of stochastic delay partial functional integro-differential equations by using the theory of resolvent operators.

The problem of controllability of nonlinear systems represented by stochastic differential equations in infinite-dimensional spaces has been extensively studied by several authors [12, 13, 16, 18, 19]. Moreover, the controllability of neutral stochastic integro-differential systems is currently an untreated topic in the literature. Motivated by this fact, the main aim of this paper is to focus on the controllability for the following neutral stochastic delay partial functional integro-differential equations perturbed by a fractional Brownian motion:

$$\begin{cases} d[x(t)+G(t,x(t-r(t)))] & = [A(t)x(t)+G(t,x(t-r(t)))+Hu(t)]dt \\ & + \int_0^t B(t-s)[x(s)+G(s,x(s-r(s)))]dsdt \\ & +F(t,x(t-\rho(t)))dt+\sigma(t)dB^H(t),\ t\in[0,T], \\ x(t)=\varphi(t),\ -\tau\le t\le 0. & \end{cases} \tag{1.1}$$

Here, $A(t)$ are closed linear operators on a separable Hilbert space X with dense domain $D(A)$ which is independent of t, and $B(t)$ are closed linear operators on X with domain $D(B(t))\supset D(A)$. The control function $u(.)$ takes values in $L^2([0,T],U)$, the Hilbert space of admissible control functions for a separable Hilbert space U. The symbol H stands for a bounded linear operator from U into X. B^H is a Fractional Brownian motion on a real and separable Hilbert space Y;

$$r,\ \rho:[0,+\infty)\to[0,\tau],\ (\tau>0)$$

are continuous; and

$$F,G:[0,+\infty)\times X\to X,\ \ \sigma:[0,+\infty)\to\mathcal{L}_2^0(Y,X)$$

are appropriate functions. Here $\mathcal{L}_2^0(Y,X)$ denotes the space of all Q-Hilbert-Schmidt operators from Y into X (see section 2 below).

In this paper, we study the controllability result with the help of resolvent operators. The resolvent operator is similar to the evolution operator for nonautonomous differential equations in a Hilbert spaces. It will not, however, be an evolution operator because it will not satisfy an evolution or semigroup property. On the other hand, to the best of our knowledge, there is no paper which investigates the controllability of neutral stochastic integro-differential equations with delays driven by a fractional Brownian motion . Thus, we will make the first attempt to study such problem in this paper.

The rest of this paper is organized as follows, In Section 2, we introduce some notations, concepts, and basic results about fractional Brownian motion and Wiener integrals over Hilbert spaces, and we mention a few results and notations related to resolvent of operators. In Section 3, the controllability of the system (1.1) is investigated via a fixed-point analysis approach. An example presented in Section 4 demonstrates the controllability result of section 3.

2. Preliminaries

In this section, we give some basic definitions and results about fractional Brownian motion and resolvant operators. For details of this section, we refer the reader to [15, 8] and the references therein.

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ be a complete probability space satisfying the usual condition, which means that the filtration is a right continuous increasing family and $\mathcal{F}_0$ contains all P-null sets.

Consider a time interval $[0, T]$ with arbitrary fixed horizon T and let $\{\beta^H(t), t \in [0, T]\}$ be a one-dimensional fractional Brownian motion with Hurst parameter $H \in (1/2, 1)$. This means by definition that β^H is a centered Gaussian process with covariance function

$$R_H(s,t) = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H}).$$

Moreover, β^H has the following Wiener integral representation

$$\beta^H(t) = \int_0^t K_H(t,s)d\beta(s), \tag{2.1}$$

where $\beta = \{\beta(t) : t \in [0, T]\}$ is a Wiener process, and $K_H(t; s)$ is the kernel given by

$$K_H(t,s) = c_H s^{\frac{1}{2}-H} \int_s^t (u-s)^{H-\frac{3}{2}} u^{H-\frac{1}{2}} du,$$

for $t > s$, where $c_H = \sqrt{\frac{H(2H-1)}{\beta(2-2H, H-\frac{1}{2})}}$ and $\beta(,)$ denotes the Beta function. We put $K_H(t,s) = 0$ if $t \leq s$.

We will denote by $\mathcal{H}$ the reproducing kernel Hilbert space of the fBm. In fact, $\mathcal{H}$ is the closure of set of indicator functions $\{1_{[0;t]}, t \in [0, T]\}$ with respect to the scalar product

$$\langle 1_{[0,t]}, 1_{[0,s]} \rangle_{\mathcal{H}} = R_H(t,s).$$

The mapping $1_{[0,t]} \to \beta^H(t)$ can be extended to an isometry between $\mathcal{H}$ and the first Wiener chaos and we will denote by $\beta^H(\varphi)$ the image of φ by the previous isometry.

We recall that for $\psi, \varphi \in \mathcal{H}$ their scalar product in $\mathcal{H}$ is given by

$$\langle \psi, \varphi \rangle_{\mathcal{H}} = H(2H-1) \int_0^T \int_0^T \psi(s)\varphi(t)|t-s|^{2H-2} ds dt.$$

Let us consider the operator K_H^* from $\mathcal{H}$ to $L^2([0, T])$ defined by

$$(K_H^* \varphi)(s) = \int_s^T \varphi(r) \frac{\partial K}{\partial r}(r,s) dr.$$

We refer to [15] for the proof of the fact that K_H^* is an isometry between $\mathcal{H}$ and $L^2([0, T])$. Moreover, for any $\varphi \in \mathcal{H}$, we have

$$\beta^H(\varphi) = \int_0^T (K_H^* \varphi)(t) d\beta(t).$$

It follows from [15] that the elements of $\mathcal{H}$ may be not functions but rather distributions of negative order. In order to obtain a space of functions contained in $\mathcal{H}$, we consider the linear space $|\mathcal{H}|$ generated by the measurable functions ψ such that

$$\|\psi\|^2_{|\mathcal{H}|} := \alpha_H \int_0^T \int_0^T |\psi(s)||\psi(t)||s-t|^{2H-2} ds dt < \infty,$$

where $\alpha_H = H(2H-1)$. We have the following Lemma (see [15]):

Lemma 2.1. *The space $|\mathcal{H}|$ is a Banach space with the norm $\|\psi\|_{|\mathcal{H}|}$ and the following inclusions hold:*

$$\mathbb{L}^2([0,T]) \subseteq \mathbb{L}^{1/H}([0,T]) \subseteq |\mathcal{H}| \subseteq \mathcal{H}.$$

Also, for any $\varphi \in \mathbb{L}^2([0,T])$,

$$\|\psi\|^2_{|\mathcal{H}|} \leq 2HT^{2H-1} \int_0^T |\psi(s)|^2 ds.$$

Let X and Y be two real, separable Hilbert spaces and let $\mathcal{L}(Y,X)$ be the space of bounded linear operator from Y to X. For convenience, we shall use the same notation to denote the norms in X, Y, and $\mathcal{L}(Y,X)$. Let $Q \in \mathcal{L}(Y,Y)$ be an operator defined by $Qe_n = \lambda_n e_n$ with finite trace $trQ = \sum_{n=1}^{\infty} \lambda_n < \infty$, where $\lambda_n \geq 0$ $(n = 1, 2, ...)$ are non-negative real numbers and $\{e_n\}$ $(n = 1, 2...)$ is a complete orthonormal basis in Y. Let $B^H = (B^H(t))$ be $Y-$ valued fBm on $(\Omega, \mathcal{F}, \mathbb{P})$ with covariance Q defined by:

$$B^H(t) = B^H_Q(t) = \sum_{n=1}^{\infty} \sqrt{\lambda_n} e_n \beta^H_n(t),$$

where β^H_n are real, independent fBm's. This process is Gaussian, it starts from 0, has zero mean and covariance:

$$E\langle B^H(t), x\rangle \langle B^H(s), y\rangle = R(s,t)\langle Q(x), y\rangle \text{ for all } x, y \in Y \text{ and } t, s \in [0,T].$$

In order to define Wiener integrals with respect to the Q-fBm, we introduce the space $\mathcal{L}^0_2 := \mathcal{L}^0_2(Y,X)$ of all Q-Hilbert-Schmidt operators $\psi : Y \to X$. We recall that $\psi \in \mathcal{L}(Y,X)$ is called a Q-Hilbert-Schmidt operator if

$$\|\psi\|^2_{\mathcal{L}^0_2} := \sum_{n=1}^{\infty} \|\sqrt{\lambda_n}\psi e_n\|^2 < \infty,$$

and the space $\mathcal{L}^0_2$ equipped with the inner product $\langle \varphi, \psi\rangle_{\mathcal{L}^0_2} = \sum_{n=1}^{\infty} \langle \varphi e_n, \psi e_n\rangle$ is a separable Hilbert space.

Now, let $\phi(s)$; $s \in [0,T]$ be a function with values in $\mathcal{L}^0_2(Y,X)$, The Wiener integral of ϕ with respect to B^H is defined by

$$\int_0^t \phi(s) dB^H(s) = \sum_{n=1}^{\infty} \int_0^t \sqrt{\lambda_n}\phi(s) e_n d\beta^H_n(s) = \sum_{n=1}^{\infty} \int_0^t \sqrt{\lambda_n}(K^*_H(\phi e_n)(s) d\beta_n(s), \tag{2.2}$$

where β_n is the standard Brownian motion used to present β^H_n as in (2.1).

We end this subsection by stating the following result which is critical in the proof of our result. It can be proved by similar arguments as those used to prove Lemma 2 in [5].

Lemma 2.2. *If $\psi : [0,T] \to \mathcal{L}_2^0(Y,X)$ satisfies $\int_0^T \|\psi(s)\|^2_{\mathcal{L}_2^0} ds < \infty$ then the sum in* (2.2) *is well-defined as an X-valued random variable, and*

$$\mathbb{E}\|\int_0^t \psi(s)dB^H(s)\|^2 \leq 2Ht^{2H-1}\int_0^t \|\psi(s)\|^2_{\mathcal{L}_2^0} ds.$$

Before proceeding to the main result, we shall make the following assumptions [8]:

($\mathcal{A}_1$) $A(t)$ generates a strongly continuous semigroup of evolution operators in the Banach space X.

($\mathcal{A}_2$) Suppose that Y represents the Banach space $D(A)$ equipped with the graph norm. $A(t)$ and $B(t)$ are closed operators; it follows that $A(t)$ and $B(t)$ are in the set $\mathcal{B}(Y,X)$ of bounded operator from Y to X for $0 \leq t \leq T$. Further, $A(t)$ and $B(t)$ are continuous on $0 \leq t \leq T$ into $\mathcal{B}(Y,X)$.

Definition 2.3. A resolvent operator for (1.1) is a bounded operator valued function $R(t,s) \in \mathcal{B}(X)$, the space of bounded linear operators on X, $0 \leq s \leq t \leq T$, having the following properties:

(a) $R(t,s)$ is strongly continuous in s and t , $R(s,s)x = x$ for all $x \in X$, $\|R(s,t)\| \leq Me^{\beta(t-s)}$ for some constants M and β.
(b) $R(t,s)Y \subseteq Y$, $R(t,s)$ is strongly continuous in s and t on Y.
(c) For $x \in X$, $R(t,s)x$ is continuously differentiable in t and s and

$$\frac{\partial R(t,s)x}{\partial t} = A(t)R(t,s)x + \int_s^t B(t-\tau)R(\tau,s)xd\tau, \tag{2.3}$$

$$\frac{\partial R(t,s)x}{\partial s} = -R(t,s)A(s)x - \int_s^t R(t-\tau)B(\tau-s)xd\tau, \tag{2.4}$$

with $\frac{\partial R(t,s)x}{\partial t}$ and $\frac{\partial R(t,s)x}{\partial s}$ are strongly continuous on $0 \leq s \leq t \leq T$. Here $R(t,s)$ can be extracted from the evolution operator of the generator $A(t)$. More details about resolvent operator can be found in [8].

3. Controllability Result

In this section, we present and prove the controllability results for the system (1.1). Before starting, we introduce the concept of a mild solution of the problem (1.1) and controllability of neutral integro-differential stochastic functional differential equations. Motivated by the theory of resolvent operators, we introduce the following concept of mild solution for equation (1.1).

Definition 3.1. *An X-valued stochastic process $\{x(t) : \ t \in [-\tau,T]\}$, is called a mild solution of equation (1.1) if*

i) $x(.) \in \mathcal{C}([-\tau,T], \mathbb{L}^2(\Omega,X))$,
ii) $x(t) = \varphi(t)$, $-\tau \leq t \leq 0$.
iii) For arbitrary $t \in [0,T]$, we have

$$\begin{aligned} x(t) &= R(t,0)(\varphi(0) + G(0,\varphi(-r(0)))) - G(t,x(t-r(t))) \\ &+ \int_0^t R(t,s)[Hu(s) + F(s,x(s-\rho(s)))]ds \\ &+ \int_0^t R(t,s)\sigma(s)dB^H(s) \quad \mathbb{P}-a.s. \end{aligned} \tag{3.1}$$

Definition 3.2. *The system (1.1) is said to be controllable on the interval* $[-\tau, T]$*, if for every initial stochastic process* $\varphi \in \mathcal{C}([-\tau, 0], \mathbb{L}^2(\Omega, X))$*, there exists a stochastic control* $u \in L^2([0,T], U)$ *such that the mild solution* $x(.)$ *of (1.1) satisfies* $x(T) = x_1$*, where* $x_1 \in \mathbb{L}^2(\Omega, X)$ *and* T *are the preassigned terminal state and time, respectively.*

Roughly speaking, the controllability problem for evolution system consists in driving the state of the system (the mild solution of the controlled equation under consideration) from an arbitrary initial state to an arbitrary final state in finite time.

To prove the controllability result, we consider the following assumptions:

($\mathcal{H}$.1) The resolvent operator $(R(t,s))_{0\le s\le t\le T}$ given by Definition 2.3 satisfies the following condition: there is a positive constant M such that

$$\sup_{0\le s,t\le T} \|R(t,s)\| \le M.$$

($\mathcal{H}$.2) The function $f : [0, +\infty) \times X \to X$ satisfies the following growth conditions: that is, there exist positive constants $C_i := C_i(T), i = 1, 2$ such that, for all $t \in [0,T]$ and $x, y \in X$

(i) $\|F(t,x) - F(t,y)\| \le C_1 \|x - y\|$,

(ii) $\|F(t,x)\|^2 \le C_2(1 + \|x\|^2)$.

($\mathcal{H}$.3) The function $G : [0, +\infty) \times X \longrightarrow X$ satisfies the following growth conditions: there exist positive constants C_3 and C_4, $C_3 < \frac{1}{2}$, such that, for all $t \in [0,T]$ and $x, y \in X$

(i) $\|G(t,x) - G(t,y)\| \le C_3 \|x - y\|$,

(ii) $\|G(t,x)\|^2 \le C_4(1 + \|x\|^2)$.

($\mathcal{H}$.4) The function G is continuous in the quadratic mean sense:

$$\text{For all } \ x \in \mathcal{C}([0,T], \mathbb{L}^2(\Omega, X)), \ \ \lim_{t\to s} \mathbb{E}\|G(t, x(t)) - G(s, x(s))\|^2 = 0.$$

($\mathcal{H}$.5) The function $\sigma : [0, +\infty) \to \mathcal{L}_2^0(Y, X)$ satisfies

$$\int_0^T \|\sigma(s)\|^2_{\mathcal{L}_2^0} ds < \infty, \ \ \forall T > 0.$$

($\mathcal{H}$.6) The linear operator W from U into X defined by

$$Wu = \int_0^T R(T,s)Hu(s)ds$$

has an inverse operator W^{-1} that takes values in $L^2([0,T], U) \setminus kerW$, where $kerW = \{x \in L^2([0,T], U), \ Wx = 0\}$ (see [9]), and there exists finite positive constants M_h, M_w such that $\|H\| \le M_h$ and $\|W^{-1}\| \le M_w$.

Moreover, we assume that $\varphi \in \mathcal{C}([-\tau, 0], \mathbb{L}^2(\Omega, X))$.

We can now state the main result of this paper.

Theorem 3.3. *Suppose that* $(\mathcal{H}.1) - (\mathcal{H}.6)$ *hold. Then, the system (1.1) has a unique mild solution on* $[-\tau, T]$ *which satisfies* $x(T) = x_1$*. Thus, the system (1.1) is controllable on* $[-\tau, T]$*.*

Proof. Fix $T > 0$ and let $\mathcal{B}_T := \mathcal{C}([-\tau, T], \mathbb{L}^2(\Omega, X))$ be the Banach space of all continuous functions from $[-\tau, T]$ into $\mathbb{L}^2(\Omega, X)$, equipped with the supremum norm $\|\xi\|_{\mathcal{B}_T} = \sup_{u\in[-\tau,T]} \left(\mathbb{E}\|\xi(u)\|^2\right)^{1/2}$ and let us consider the set

$$S_T = \{x \in \mathcal{B}_T : x(s) = \varphi(s), \text{ for } s \in [-\tau, 0]\}.$$

S_T is a closed subset of $\mathcal{B}_T$ provided with the norm $\|.\|_{\mathcal{B}_T}$.

Using the hypothesis $(\mathcal{H}.6)$ for an arbitrary function $x(.)$, define the control

$$\begin{aligned} u(t) &= W^{-1}\{x_1 - R(T,0)(\varphi(0) + G(0, \varphi(-r(0)))) + G(T, x(T - r(T))) \\ &- \int_0^T R(T,s)F(s, x(s - \rho(s)))ds - \int_0^T R(T,s)\sigma(s)dB^H(s)\}(t). \end{aligned} \tag{3.2}$$

We shall now show that when using this control, the operator Φ defined on S_T by $\Phi(x)(t) = \varphi(t)$ for $t \in [-\tau, 0]$ and for $t \in [0, T]$

$$\begin{aligned} \Phi(x)(t) &= R(t,0)(\varphi(0) + G(0, \varphi(-r(0)))) - G(t, x(t - r(t))) \\ &+ \int_0^t R(t,s)[Hu(s) + F(s, x(s - \rho(s)))]ds] + \int_0^t R(t,s)\sigma(s)dB^H(s) \end{aligned} \tag{3.3}$$

has a fixed point. Substituting (3.2) in (3.3) we can show that $(\psi x)(T) = x_1$, which means that the control u steers the system from the initial state φ to x_1 in time T, provided we can obtain a fixed point of the operator ψ which implies that the system in controllable.

Next, we will show by using Banach fixed point theorem that ψ has a unique fixed point. We divide the subsequent proof into two steps.

Step 1: For arbitrary $x \in S_T$, let us prove that $t \to \Phi(x)(t)$ is continuous on the interval $[0, T]$ in the $\mathbb{L}^2(\Omega, X)$-sense.

Let $0 < t < T$ and $|h|$ be sufficiently small. Then, for any fixed $x \in S_T$, we have

$$\begin{aligned} \mathbb{E}\|\Phi(x)(t &+ h) - \Phi(x)(t)\|^2 \leq 5\mathbb{E}\|(R(t+h,0) - R(t,0))(\varphi(0) + G(0, \varphi(-r(0))))\|^2 \\ &+ 5\mathbb{E}\|G(t+h, x(t+h - r(t+h))) - G(t, x(t - r(t)))\|^2 \\ &+ 5\mathbb{E}\|\int_0^{t+h} R(t+h,s)F(s - \rho(s))ds - \int_0^t R(t,s)F(s - \rho(s))ds\|^2 \\ &+ 5\mathbb{E}\|\int_0^{t+h} R(t+h,s)\sigma(s)dB^H(s) - \int_0^t R(t,s)\sigma(s)dB^H(s)\|^2 \\ &+ 5\mathbb{E}\|\int_0^{t+h} R(t+h,\nu)HW^{-1}\{x_1 - R(T,0)(\varphi(0) + G(0, \varphi(-r(0)))) \\ &+ G(T, x(T - r(T))) - \int_0^T R(T,s)F(s, x(s - \rho(s)))ds \\ &- \int_0^T R(T,s)\sigma(s)dB^H(s)\}d\nu - \int_0^t R(t,\nu)HW^{-1}\{x_1 - R(T,0)(\varphi(0) \\ &+ G(0, \varphi(-r(0)))) + G(T, x(T - r(T))) \end{aligned}$$

$$\begin{aligned} &- \int_0^T R(T,s)F(s,x(s-\rho(s)))ds - \int_0^T R(T,s)\sigma(s)dB^H(s)\}d\nu\|^2 \\ &= \sum_{1\le i\le 5} 5\mathbb{E}\|I_i(h)\|^2. \end{aligned}$$

We are going to show that each function $t \to I_i(t)$ is continuous on $[0,T]$ in the $\mathbb{L}^2$-sense.
By the strong continuity of $R(t,s)$, we have

$$\lim_{h\to 0}(R(t+h,0)-R(t,0))(\varphi(0)+G(0,\varphi(-r(0)))) = 0.$$

The condition $(\mathcal{H}.1)$ guarantees that

$$\|(R(t+h,0)-R(t,0))(\varphi(0)+G(0,\varphi(-r(0))))\| \le 2M\|\varphi(0)+G(0,\varphi(-r(0)))\| \in \mathbb{L}^2(\Omega).$$

Then we conclude by the Lebesgue dominated convergence theorem that

$$\lim_{h\to 0}\mathbb{E}\|I_1(h)\|^2 = 0.$$

By using the fact that the operator G is continuous in the quadratic mean sense, we conclude by condition $(\mathcal{H}.4)$ that

$$\lim_{h\to 0}\mathbb{E}\|I_2(h)\|^2 = 0.$$

For the third term $I_3(h)$, we suppose that $h > 0$, note that similar estimates hold for $h < 0$. Then, we have

$$\begin{aligned} \|I_3(h)\| &\le \|\int_0^t (R(t+h,s)-R(t,s))F(s,x(s-r(s)))ds\| \\ &\quad +\|\int_t^{t+h} R(t+h,s)F(s,x(s-r(s)))ds\| \\ &\le I_{31}(h)+I_{32}(h). \end{aligned}$$

By Hölder's inequality, we obtain

$$\mathbb{E}|I_{31}(h)|^2 \le t\mathbb{E}\int_0^t \|(R(t+h,s)-R(t,s))F(s,x(s-r(s)))\|^2 ds$$

By using the strong continuity of $R(t,s)$, we have for each $s\in[0,t]$,

$$\lim_{h\to 0}(R(t+h,s)-R(t,s))F(s,x(s-r(s))) = 0.$$

By using condition $(\mathcal{H}.1)$ and condition (ii) in $(\mathcal{H}.2)$, we obtain

$$\|(R(t+h,s)-R(t,s))F(s,x(s-r(s)))\|^2 \le 4M^2\|F(s,x(s-r(s)))\|^2.$$

So, we conclude by the Lebesgue dominated convergence theorem that

$$\lim_{h\to 0}\mathbb{E}\|I_{31}(h)\|^2 = 0.$$

By conditions $(\mathcal{H}.1)$, $(\mathcal{H}.2)$ and Hölder's inequality, we get

$$\mathbb{E}\|I_{32}(h)\|^2 \le C_2^2 hM^2\int_0^T (\mathbb{E}\|x(s-r(s))\|^2+1)ds.$$

So that

$$\lim_{h\to 0}\mathbb{E}\|I_3(h)\|^2 = 0.$$

For the term $I_4(h)$, we have

$$\begin{aligned}\|I_4(h)\| &\leq \|\int_0^t (R(t+h,s)-R(t,s))\sigma(s)dB^H(s)\| \\ &+ \|\int_t^{t+h} R(t+h,s)\sigma(s)dB^H(s)\| \\ &\leq I_{41}(h)+I_{42}(h).\end{aligned}$$

By condition $(\mathcal{H}.1)$ and Lemma 2.2, we get that

$$E|I_{41}(h)|^2 \leq 2Ht^{2H-1}\int_0^t \|(R(t+h,s)-R(t,s))\sigma(s)\|^2_{\mathcal{L}_2^0}ds.$$

Since $\lim\limits_{h\to 0}\|(R(t+h,s)-R(t,s))\sigma(s)\|^2_{\mathcal{L}_2^0}=0$ and

$$\|(R(t+h,s)-R(t,s))\sigma(s)\|^2_{\mathcal{L}_2^0} \leq 4M^2\|\sigma(s)\|^2_{\mathcal{L}_2^0} \in \mathbb{L}^1([0,T],ds),$$

we conclude, by the Lebesgue dominated convergence theorem that,

$$\lim_{h\to 0}\mathbb{E}|I_{41}(h)|^2=0.$$

Again by Lemma 2.2, we get that

$$\mathbb{E}|I_{42}(h)|^2 \leq 2Hh^{2H-1}M^2\int_t^{t+h}\|\sigma(s)\|^2_{\mathcal{L}_2^0}ds \to 0.$$

Next, let's observe that

$$\begin{aligned}\mathbb{E}\|I_5(h)\|^2 &\leq 2\mathbb{E}\|\textstyle\int_t^{t+h} R(t+h,\nu)HW^{-1}\{x_1-R(T,0)(\varphi(0)+G(0,\varphi(-r(0))))\\ &+G(T,x(T-r(T)))-\textstyle\int_0^T R(T,s)F(s,x(s-\rho(s)))ds\\ &-\textstyle\int_0^T R(T,s)\sigma(s)dB^H(s)\}d\nu\|^2\\ &+2\mathbb{E}\|\textstyle\int_0^t(R(t+h,\nu)-R(t,\nu))HW^{-1}\{x_1-R(T,0)(\varphi(0)\\ &+G(0,\varphi(-r(0))))+G(T,x(T-r(T)))\\ &-\textstyle\int_0^T R(T,s)F(s,x(s-\rho(s)))ds-\textstyle\int_0^T R(T,s)\sigma(s)dB^H(s)\}d\nu\|^2\\ &\leq 2[\mathbb{E}\|I_{5,1}(h)\|^2+\mathbb{E}\|I_{5,2}(h)\|^2].\end{aligned}$$

Let's first deal with $I_{5,1}(h)$. Using conditions $(\mathcal{H}.1)-(\mathcal{H}.6)$ and Hölder inequality, it follows that

$$\begin{aligned}\mathbb{E}\|I_{5,1}(h)\|^2 &\leq 5M^2M_h^2M_w^2\textstyle\int_t^{t+h}\{\mathbb{E}\|x_1\|^2+M^2\mathbb{E}\|\varphi(0)+G(0,\varphi(-r(0)))\|^2\\ &+C_4^2(1+\sup_{s\in[-\tau,T]}\mathbb{E}\|x(s)\|^2)+M^2TC_2^2(1+\sup_{s\in[-\tau,T]}\mathbb{E}\|x(s)\|^2)\\ &+2M^2HT^{2H-1}\textstyle\int_0^T\|\sigma(s)\|^2_{\mathcal{L}_2^0}ds\}d\nu.\end{aligned}$$

Assures

$$\lim_{h\to 0}\mathbb{E}||I_{5,1}(h)||^2=0.$$

In a similar way, we have

$$\begin{aligned}\mathbb{E}\|I_{5,2}(h)\|^2 &\le 5M_h^2M_w^2\int_0^t \|(R(t+h,\nu)-R(t,\nu))\|^2\{\mathbb{E}\|x_1\|^2\\ &+M^2\mathbb{E}\|\varphi(0)+G(0,\varphi(-r(0)))\|^2\\ &+C_4^2(1+\sup_{s\in[-\tau,T]}\mathbb{E}\|x(s)\|^2)\\ &+M^2T^2C_2^2(1+\sup_{s\in[-\tau,T]}\mathbb{E}\|x(s)\|^2)\\ &+2M^2HT^{2H-1}\int_0^T\|\sigma(s)\|^2_{\mathcal{L}_2^0}ds\}d\nu.\end{aligned}$$

Since

$$\begin{aligned}&\|R(t+h,\nu)-R(t,\nu)\|^2\{\mathbb{E}\|x_1\|^2+M^2\mathbb{E}\|\varphi(0)+G(0,\varphi(-r(0)))\|^2\\ &+C_4^2(1+\sup_{s\in[-\tau,T]}\mathbb{E}\|x(s)\|^2)+M^2T^2C_2^2(1+\sup_{s\in[-\tau,T]}\mathbb{E}\|x(s)\|^2)\\ &+2M^2HT^{2H-1}\int_0^T\|\sigma(s)\|^2_{\mathcal{L}_2^0}ds\}\\ &\le 4M^2\{\mathbb{E}\|x_1\|^2+M^2\mathbb{E}\|\varphi(0)+G(0,\varphi(-r(0)))\|^2\\ &+C_4^2(1+\sup_{s\in[-\tau,T]}\mathbb{E}\|x(s)\|^2)\\ &+M^2T^2C_2^2(1+\sup_{s\in[-\tau,T]}\mathbb{E}\|x(s)\|^2)\\ &+2M^2HT^{2H-1}\int_0^T\|\sigma(s)\|^2_{\mathcal{L}_2^0}ds\}\in\mathbb{L}^1([0,T],ds]),\end{aligned}$$

we conclude, by the Lebesgue dominated convergence theorem that

$$\lim_{h\to 0}\mathbb{E}||I_{5,2}(h)||^2=0.$$

The above arguments show that $\lim_{h\to 0}\mathbb{E}\|\Phi(x)(t+h)-\Phi(x)(t)\|^2=0$. Hence, we conclude that the function $t\to\Phi(x)(t)$ is continuous on $[0,T]$ in the $\mathbb{L}^2$-sense.

Step 2. Now, we are going to show that Φ is a contraction mapping in S_{T_1} with some $T_1\le T$ to be specified later.

Let $x,y\in S_T$ we obtain for any fixed $t\in[0,T]$

$$\begin{aligned}&\|\Phi(x)(t)-\Phi(y)(t)\|^2\le\\ &4\|G(t,x(t-r(t)))-G(t,y(t-r(t)))\|^2\\ &+4\|\int_0^t R(t,s)[F(s,x(s-\rho(s)))-F(s,y(s-\rho(s)))]ds\|^2\\ &+4\|\int_0^t R(t,\nu)HW^{-1}[G(T,x(T-r(T)))-G(T,y(T-r(T)))]d\nu\|^2\end{aligned}$$

$$+4\|\int_0^t R(t,\nu)HW^{-1}\int_0^T R(T,s)[F(s,x(s-\rho(s)))-F(s,y(s-\rho(s)))]dsd\nu\|^2.$$

Using growth conditions on F and G combined with Hölder's inequality, we obtain

$$\begin{aligned}\mathbb{E}\|\Phi(x)(t)-\Phi(y)(t)\|^2 &\leq 4C_3^2\mathbb{E}\|x(t-r(t))-y(t-r(t))\|^2\\ &+4M^2C_1^2t\int_0^t \mathbb{E}\|x(s-r(s))-y(s-r(s))\|^2ds\\ &+4\mathbf{t}M^2M_h^2M_w^2[\mathbb{E}\|x(T-r(T))-y(T-r(T))\|^2\\ &+T^2C_1^2M^2\sup_{s\in[-\tau,t]}\mathbb{E}\|x(s)-y(s)\|^2.\end{aligned}$$

Hence,

$$\sup_{s\in[-\tau,t]}\mathbb{E}\|\Phi(x)(s)-\Phi(y)(s)\|^2\leq\gamma(t)\sup_{s\in[-\tau,t]}\mathbb{E}\|x(s)-y(s)\|^2,$$

where

$$\gamma(t)=4[C_3^2+M^2C_1^2t^2+tM^2M_h^2M_w^2(1+T^2C_1^2M^2)].$$

By condition (iii) in $(\mathcal{H}.3)$, we have $\gamma(0)=4C_3^2<1$. Then there exists $0<T_1\leq T$ such that $0<\gamma(T_1)<1$ and Φ is a contraction mapping on S_{T_1} and therefore has a unique fixed point, which is a mild solution of equation (1.1) on $[-\tau,T_1]$. This procedure can be repeated in order to extend the solution to the entire interval $[-\tau,T]$ in finitely many steps. Clearly, $(\psi x)(T)=x_1$ which implies that the system (1.1) is controllable on $[-\tau,T]$. This completes the proof. □

4. Example

We consider the following stochastic partial neutral functional integro-differential equation with finite delays τ_1 and τ_2 $(0\leq\tau_i\leq\tau<\infty,\ i=1,2)$, driven by a fractional Brownian motion of the form

$$\begin{cases}\frac{\partial}{\partial t}[x(t,\xi)+g(t,x(t-\tau_1,\xi))]=\frac{\partial^2}{\partial^2\xi}[x(t,\xi)+g(t,x(t-\tau_1,\xi))]\\ +\int_0^t b(t-s)\frac{\partial^2}{\partial^2\xi}[x(s,\xi)+g(s,x(s-\tau_1,\xi))]ds\\ +f(t,x(t-\tau_2,\xi))+\mu(t,\xi)+\sigma(t)\frac{dB^H}{dt}(t), \qquad t\in J:=[0,1],\\ x(t,0)+g(t,x(t-\tau_1,0))=0, \qquad t\geq 0,\\ x(t,1)+g(t,x(t-\tau_1,1))=0, \quad t\geq 0,\\ x(s,\xi)=\varphi(s,\xi), \ -\tau\leq s\leq 0 \quad a.s.,\end{cases} \tag{4.1}$$

where $B^H(t)$ is a fractional Brownian motion, $f,g:\mathbb{R}^+\times\mathbb{R}\longrightarrow\mathbb{R}$ and $b:\mathbb{R}^+\longrightarrow\mathbb{R}$ are continuous functions and $\varphi:[-\tau,0]\times[0,1]\longrightarrow\mathbb{R}$ is a given continuous function such that $\varphi(s,.)\in L^2([0,1])$ is measurable and satisfies $\mathbb{E}\|\varphi\|^2<\infty$.

We rewrite (4.1) in the abstract form of (1.1). Let $X = \mathbb{L}^2(J)$ and define $A(t) : X \longrightarrow X$ by $A(t)z = a(t,y)z"$ with domain

$$D(A) = \{z \in X : z \text{ and } z' \text{ are absolutely continuous, } z" \in X, z(0) = z(1) = 0\}.$$

This family of operators generates an evolution system and $R(t,s)$ can be extracted from the evolution system [8] such that $|R(t,s)| \leq M$, $M > 0$, for $s < t$.
We assume that the following conditions hold:

(i) Let $Hu : [0,1] \longrightarrow X$ be defined by

$$Hu(t)(\xi) = \mu(t,\xi),\ 0 \leq \xi \leq 1,\ u \in L^2([0,1],U).$$

(ii) Assume that the operator $W : L^2([0,1],U) \longrightarrow X$ given by

$$Wu(\xi) = \int_0^T R(T,s)\mu(t,\xi)ds, \quad 0 \leq \xi \leq 1,$$

has a bounded invertible operator W^{-1} and satisfies condition $(\mathcal{H}.6)$. (For the construction of the operator W and its inverse, see [17]).

(iii) For $t \in [0,1]$, $f(t,0) = g(t,0) = 0$,

(iv) There exist positive constants C_1, and C_3, $C_3 < \frac{1}{2}$, such that

$$|f(t,\xi_1) - f(t,\xi_2)| \leq C_1|\xi_1 - \xi_2|, \text{ for } t \in [0,1] \text{ and } \xi_1,\xi_2 \in \mathbb{R},$$
$$|g(t,\xi_1) - g(t,\xi_2)| \leq C_3|\xi_1 - \xi_2|, \text{ for } t \in [0,1] \text{ and } \xi_1,\xi_2 \in \mathbb{R},$$

(v) There exist positive constants C_2 and C_4, such that

$$|f(t,\xi)| \leq C_2(1 + |\xi|^2), \text{ for } t \in [0,1] \text{ and } \xi \in \mathbb{R},$$
$$|g(t,\xi)| \leq C_4(1 + |\xi|^2), \text{ for } t \in [0,1] \text{ and } \xi \in \mathbb{R},$$

(vi) The function $\sigma : [0,+\infty) \to \mathcal{L}_2^0(L^2([0,1]), L^2([0,1]))$ satisfies

$$\int_0^1 \|\sigma(s)\|^2_{\mathcal{L}_2^0} ds < \infty.$$

Define the operators $F, G : \mathbb{R}^+ \times L^2([0,1]) \longrightarrow L^2([0,1])$ by

$$F(t,\phi)(\xi) = f(t,\phi(-\tau_1)(\xi)) \text{ for } \xi \in [0,1] \text{ and } \phi \in L^2([0,1]),$$

and

$$G(t,\phi)(\xi) = g(t,\phi(-\tau_2)(\xi)), \text{ and } \phi \in L^2([0,1])$$

If we put

$$\begin{cases} x(t)(\zeta) = x(t,\zeta),\ t \in [0,1] \text{ and } \zeta \in [0,1] \\ x(t,\zeta) = \varphi(t,\zeta),\ t \in [-\tau,0] \text{ and } \zeta \in [0,1], \end{cases} \tag{4.2}$$

then, the problem (4.1) can be written in the abstract form

$$\begin{cases} d[x(t) + G(t,x(t-r(t)))] & = [Ax(t) + G(t,x(t-r(t)))]dt + \int_0^t B(t-s)[x(s) \\ & +G(s,x(s-r(s)))]dsdt + [F(t,x(t-\rho(t))) + Hu(t)]dt \\ & +\sigma(t)dB^H(t),\ 0 \leq t \leq 1, \\ x(t) = \varphi(t),\ -\tau \leq t \leq 0. \end{cases}$$

As a consequence of the continuity of f and g and assumption (iii) it follows that F and G are continuous. By assumption (iv), one can see that

$$\|F(t,\phi_1) - F(t,\phi_1)\|_{L^2([0,1])} \leq C_1\|\phi_1 - \phi_2\|_{L^2([0,1])},$$

$$\|G(t,\phi_1) - G(t,\phi_1)\|_{L^2([0,1])} \leq C_3\|\phi_1 - \phi_2\|_{L^2([0,1])}, \text{ with } C_3 < \frac{1}{2}.$$

Furthermore, by assumption (v), it follows that

$$\|F(t,\phi)\| \leq C_2(1 + \|\phi\|^2), \text{ for } t \in [0,1],$$
$$\|G(t,\phi)\| \leq C_4(1 + \|\phi\|^2), \text{ for } t \in [0,1].$$

all the assumptions of Theorem 3.3 are fulfilled. Therefore, we conclude that the system (4.1) is controllable on $[-\tau, 1]$.

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