8-2016

On Approximately Controlled Systems

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ON APPROXIMATELY CONTROLLABLE SYSTEMS

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The aim of this article is to provide a thorough analysis of the approximate controllability of abstract systems, beginning with the inception of the concept in the 1960s. The study begins with a discussion of abstract infinite-dimensional linear systems where the notion of approximate controllability originated. The extension of this foundational theory to nonlinear systems is reviewed, followed by a modern treatment of fractional approximate controllability which has significant applications in the mathematical modeling of phenomena across disciplines.

1. Introduction

The notion of controllability represents a major concept of modern control theory. In (Kalman, 1960) and (Kalman *et* al., 1963), Kalman introduced the concept of (*exact*) *controllability* for finite-dimensional dynamical systems. Intuitively, a dynamical system is exactly controllable if it can be steered from an arbitrary initial state to an arbitrary final state in finite time using a set of admissible controls. This work set the stage for the theory of controllability developed in the more than half a century that followed, and which was first expanded upon for abstract linear systems by H.O. Fattorini in the works (Fattorini, 1966) and (Fattorini, 1967).

In the decade that followed, researchers investigated this notion of controllability for *infinite-dimensional* abstract systems and encountered an interesting finding: for abstract linear systems taking values in an infinite-dimensional, separable Banach space, under conditions comparable to those established in the finite-dimensional theory, the system could *not* be exactly controllable on any finite interval. This observation was made by R. Triggiani in (Triggiani, 1972, 1975). This prompted the formulation of the weaker notion of *approximate* controllability. Intuitively, a dynamical system is approximately controllable if an arbitrary initial state can be steered toward an arbitrarily small neighborhood of any given final state.

For infinite-dimensional systems in which one *can* formulate an exact controllability result, typically the hypotheses one must impose to establish the result are too restrictive or are extremely difficult to verify in practical situations. As such, the notion has limited applicability. However, approximate controllability results are much more prevalent and from the viewpoint of applications, approximate controllability is quite adequate. As such, studying the weaker notion of approximate controllability for nonlinear systems is not only important, it is necessary.

The remainder of the article is structured as follows. We begin in Section 2 with a brief review of foundational controllability results for abstract linear control systems. We primarily focus on the discussion provided in the seminal papers written by H.O. Fattorini (Fattorini, 1966, 1967, 1975) and R. Triggiani in the mid 1970s (Triggiani, 1972, 1975, 1976), and more recently by Bashirov and Mahmduov (Bashirov and Mahmudov, 1999). The discussion in Section 3 then focuses on some initial pinnacle contributions to the theory of approximate controllability of abstract systems formulated by equipping linear systems with a nonlinear forcing term. Among the papers reviewed are (Dauer and Mahmudov, 2002), (Mahmudov, 2003), (Naito, 1987), and (Zhou, 1983, 1984). The conditions that ensure the approximate controllability of abstract nonlinear systems are typically established using tools of semigroup theory, operator theory and fixed-point theory under the assumption that the linear part of the associated nonlinear system is, itself, approximately controllable.

The discussion and theory developed in the papers reviewed in Sections 2 and 3 paved the way for the study of approximate controllability of a diverse collection of abstract systems, including stochastic evolution equations of various types (Mahmudov and McKibben, 2006), equations of Volterra type (Ke, *et al.*, 2012), equations of Sobolev type (Kerboua, *et al.*, 2013; Mahmudov, 2013); neutral equations (Kumar and Sukavanam, 2012; Mahmudov, 2013); delay systems (Kumar and Sukavanam, 2012, 2013; Sakthivel and Ren, 2013; Sukvanam and Kumar, 2011; Yan, 2012), nonlinear differential inclusions (Sakthivel, *et al.*, 2013; Vijayakumar, *et al.*, 2013), and equations equipped with nonlocal initial conditions (Mahmudov and Zorlu, 2013), just to name a few.

The ever-expanding abstract theory developed over the past half-century is important from the viewpoint of applications, and has been applied to a continually increasing collection of concrete PDEs and nonlinear ODEs arising in the mathematical modeling of phenomena occurring in various disciplines, including physics, engineering, population ecology, etc. thereby broadening the scientific community's understanding these phenomena. The article finally concludes with a survey of very recent efforts in developing a theory of approximate controllability of abstract fractional differential equations in Section 4.

2. Approximate Controllability of Linear Systems

We begin our discussion with the seminal works of R. Triggiani (Triggiani, 1972, 1975, 1976). The theory developed in these papers is concerned with abstract control systems of the form

$$y'(t) = Ay(t) + Bu(t),\ t \geq t_0\ , \tag{0.1}$$

where $y(t)$ belongs to a complex, separable infinite-dimensional Banach space *X* (called the *state space*) with norm $\|\cdot\|_X$; *u* belongs to a complex, separable Banach space *U* (called the *control space*); $B : U \to X$ is a bounded linear operator; and the operator $A : dom(A) \subset X \to X$ satisfies the following assumption:

Assumption H2.1: $A : dom(A) \subset X \to X$ is a closed, linear densely-defined operator that is the infinitesimal generator of a strongly continuous semigroup of bounded operators $\{S(t) : t \geq t_0\}$ on *X*.

If the control space *U* is finite-dimensional with dimension *m*, say with basis $\{e_i : i = 1, ..., m\}$, (0.1) can be written as

$$y'(t) = Ay(t) + \sum_{i=1}^{m} b_i u_i\ ,\ \ t \geq t_0 \tag{0.2}$$

where $b_i(t) = B(t)e_i$ (*i* = 1, … , *m*) and u_i are the components of *u*.

The following concept of *approximate controllability* of (0.1) was introduced:

Definition 2.1

i.) System (0.1) is *approximately controllable on* $[t_0, T]$ if for every $\varepsilon > 0$ and arbitrarily chosen initial starting position y_0 and final ending position y_1 in *X*, there is an admissible control $u(t)$ on $[t_0, T]$ for which

$$\|y(T, t_0, x_0, u) - y_1\|_X \leq \varepsilon\ , \tag{0.3}$$

where $y(T, t_0, y_0, u)$ is the solution of (0.1) corresponding to the initial point $y(t_0) = y_0$ and control *u*, evaluated at time *t* = *T*. (Equivalently, for each initial starting position y_0, the set of all points to which y_0 can be steered by admissible controls on $[t_0, T]$ is dense in *X*.)

ii.) If $\varepsilon = 0$ in (i), we say that (0.1) is *exactly controllable on* $[t_0, T]$.

An *admissible control u* on $[t_0, T]$ is a *U*-valued Bochner integrable function with bounded norm $\|u(t)\|_U$ on $[t_0, T]$. For each such admissible control *u* and initial condition $y(t_0) = y_0$, it is known that under assumption H2.1, (0.1) has a unique mild solution on $[t_0, T]$ expressed by the variation of parameters formula

$$y(t, t_0, y_0, u) = S(t - t_0) y_0 + \int_{t_0}^{t} S(t - \tau) B u(\tau) d\tau \,. \qquad (0.4)$$

For brevity, we shall henceforth write $y(t)$ in place of $y(t, t_0, y_0, u)$.

Remark Definition 2.1 can be suitably modified for the more specific control system (0.2).

For the finite-dimensional case (when $X = \mathbb{R}^n$ and $U = \mathbb{R}^m$), parts (i) and (ii) of Definition 2.1 are equivalent. A well-known result in the finite-dimensional case (when $X = \mathbb{R}^n$ and $U = \mathbb{R}^m$ and so, *A* and *B* are matrices) is that (0.1) is exactly controllable (and hence, approximately controllable) if and only if $rank\left[B, AB, \dots, A^{n-1}B \right] = n$. The following characterization is the extension of this result to the infinite-dimensional case

Theorem 2.1 System (0.1) is approximately controllable on $[t_0, T]$ if and only if

$$\mathrm{cl}\left(\mathrm{span}\left\{ A^n BU : n \geq 0 \right\}\right) = X \,. \qquad (0.5)$$

Regarding system (0.2), condition (0.5) can be written as

$$\mathrm{cl}\left(\mathrm{span}\left\{ A^n b_i : n \geq 0,\ i = 1, \dots, m \right\}\right) = X \,. \qquad (0.6)$$

This result provides an algebraic test for the approximate controllability of (0.1) in contrast to the results established by Fattorini (Fattorini, 1966, 1967), which employ a much more technical apparatus involving the use of ordered representation theory of a Hilbert space for self-adjoint operators.

Triggiani discusses several examples of Volterra integral equations and specific PDEs. As part of this discussion it is shown that while some of these systems are indeed approximately controllable under appropriate conditions, they are not *exactly* controllable on any interval. As such, when *X* is infinite-dimensional there are systems of the form (0.1) that are approximately controllable on $[t_0, T]$, but not exactly controllable on $[t_0, T]$. More precisely, the following theorem is proven in (Triggiani, 1975):

Theorem 2.2 Let *X* be an infinite-dimensional Banach space. The system (0.2) defined on *X* can never be exactly controllable on any (fixed) finite interval $[t_0, T]$. The same is true for system

(0.1) defined on X under the assumptions that $B:U \to X$ is a compact operator and that X has a Schauder basis.

Consequently, the introduction of the more general concept of approximate controllability is not only nontrival, but it is essential. Even though the concept is weaker, it proves more widely applicable than exact controllability (especially considering the highly restrictive assumptions that must be imposed in order to ensure such a system in exactly controllable) and it has been shown that in practical applications, the notion of approximate controllability is more than adequate.

Remark The more general non-autonomous case in which the operators A and B depend on t is also studied in (Triggiani 1975), but for simplicity and interest of uniformity of discussion throughout the manuscript, we focus only on the autonomous case.

The above investigation is continued in the second seminal paper (Triggiani, 1976). A more specialized framework for (0.1) and (0.2) (in the sense of enhanced assumptions being imposed on the operators A and B and restricting the underlying state space X) in an effort to establish readily-verifiable criteria for the approximate controllability of these systems involving the eigenvalues of A is explored. Specifically, in addition to assumption H2.1, the following additional restrictions are imposed:

Assumption H2.2: The state space X is a Hilbert space.

Assumption H2.3: The operator $A: dom(A) \subset X \to X$ is normal and there exists μ_0 for which the resolvent operator $R(\mu_0, A)$ is a compact operator on X.

In such case, the following result is proved:

Theorem 2.3 Suppose that assumptions H2.1 – H2.3 are satisfied. Then, (0.1) is approximately controllable on $[t_0, T]$ if and only if

$$P_j(rngB) = X_j, \quad j = 1, 2, ... \tag{0.7}$$

where X_j is the r_j-dimensional eigenspace associated with the eigenvalue λ_j and P_j is the orthogonal projection of X onto X_j.

Remark The condition in Theorem 2.3 holds, for instance, when the range of the operator B is all of X.

A result concerning the approximate controllability of (0.2) is established under the following additional analyticity assumption, which holds, for instance, if A satisfies assumption H2.1 and is self-adjoint.

Assumption H2.4: The eigenvalues of *A* are contained in a sector $\Sigma = \left\{ \lambda : \left| \arg(\lambda - a) \right| \le \frac{\pi}{2} + \beta \right\}$, where *a* is a real number and $0 < \beta < \frac{\pi}{2}$.

Let $\left\{ e_{jk} : k = 1,...,r_j ,\ j = 1,2,... \right\}$ be a complete orthonormal set of eigenvectors of *A*. (Note that every *x* in *X* can be expressed uniquely as $x = \sum_{j=1}^{\infty} \sum_{k=1}^{r_j} \left\langle x, e_{jk} \right\rangle e_{jk}$.) For each *j* = 1, 2, … and every *m*-tuple $\left(b_i\right)$ of vectors in *X*, the $r_j \times m$ matrix B_j is defined as follows:

$$B_j = \begin{bmatrix} \left\langle b_1, e_{j1} \right\rangle & \cdots & \left\langle b_m, e_{j1} \right\rangle \\ \left\langle b_1, e_{j2} \right\rangle & \cdots & \left\langle b_m, e_{j2} \right\rangle \\ \vdots & \cdots & \vdots \\ \left\langle b_1, e_{j r_j} \right\rangle & \cdots & \left\langle b_m, e_{j r_j} \right\rangle \end{bmatrix} \tag{0.8}$$

The result concerning the approximate controllability of (0.2) can now be stated as follows:

Theorem 2.4 Suppose that assumptions H2.1 – H2.4 are satisfied. Then, (0.2) is approximately controllable on $[t_0, T]$ if and only if for all vectors b_i, $i = 1,...,m$ in *X*,

$$rank\ B_j = r_j \text{, for all } j = 1,\ 2,\ ... \tag{0.9}$$

Much work on the approximate controllability of abstract linear systems was published in the years following the appearance of these two papers. Of them, we next focus on the work in (Bashirov and Mahmudov, 1999). New necessary and sufficient conditions based on the convergence of a certain sequence of operators involving the resolvent of the negative of the so-called *controllability operator* are established in this paper for the approximate controllability of (0.1).

Remark The results we mention below are actually special cases of the ones proved in that paper. Indeed, a nonlinear version of (0.1) is considered, as well as a related stochastic variant. We present a simpler case, however, to highlight the nature of the conditions for approximate controllability being formulated. The nature of these conditions is novel in and of itself, and are useful for the study of controllability issues for many different types of problems.

Assume that the operator *A* satisfies assumption H2.1 and recall the variation of parameters formula (0.4). For every $0 \le t < T$, the operator $\Gamma : \left[t_0, T\right] \to \left[0, \infty\right)$ defined by

$$\Gamma(T - t) = \int_t^T S(T - s) B B^* S^* (T - s) ds,\ \ t_0 \le T \le T \tag{0.10}$$

is called a *controllability operator*.

The following theorem provides necessary and sufficient conditions for the approximate controllability of (0.1) in terms of (0.10) and the associated resolvent operators.

Theorem 2.5 The following statements are equivalent:

i.) The control system (0.1) is approximately controllable on $[t_0, T]$.

ii.) If $B^* S^*(t) x = 0$, for all $t \in [t_0, T]$, then $x = 0$.

iii.) $\lambda R(\lambda, -\Gamma(T))$ converges to the zero operator as $\lambda \to 0$ in the strong operator topology.

iv.) $\lambda R(\lambda, -\Gamma(T))$ converges to the zero operator as $\lambda \to 0$ in the weak operator topology.

Here, $R(\lambda, -\Gamma(T)) = (\lambda I - \Gamma(T))^{-1}$.

This work was continued and expanded upon in (Mahmudov, 2003).

3. Approximate Controllability of Nonlinear Systems

We now investigate the controllability of abstract systems obtained by equipping the linear control systems reviewed in Section 2 with a nonlinear forcing term. Controllability results for the nonlinear infinite dimensional case primarily concern *semilinear* control systems consisting of a linear part and a nonlinear forcing term. Various sufficient conditions for approximate controllability have been established in the past three decades.

We begin with the work of Zhou (Zhou, 1983, 1984). This work is an outgrowth of the study by Henry (Henry, 1978). Unlike Henry's results, these results apply to a nonlinear abstract control system in a Hilbert space, as well as finite-dimensional differential equations. The semilinear abstract evolution equation

$$\begin{cases} y'(t) = Ay(t) + F(y(t)) + (Bv)(t), \quad 0 < t < T \\ y(0) = y_0 \end{cases} \tag{1.1}$$

is studied. Here, $y : [0, T] \to H$ is the state trajectory and H is a Hilbert space; the control $v(\cdot)$ belongs to $L^2(0, T; V)$ (the space of square-integrable V-valued functions on $(0,T)$), where V is a possibly different Hilbert space (for instance, $V = L^2(0, T; X)$, where X is a real Hilbert space); $A : dom(A) \subset H \to H$ is a linear operator on H; $F : H \to H$ is a nonlinear operator; $B : V \to H$ is a bounded linear operator; and $y_0 \in H$.

Generally, the approximate controllability for semilinear systems is linked to the operators *B* and *F* and their relationship to one another. This fact becomes apparent when reviewing the nature of the following assumptions imposed.

Assumption H3.1: $A: dom(A) \subset H \to H$ is a linear operator that generates a differentiable semigroup $\{S(t): 0 \le t \le T\}$ on *H*.

Assumption H3.2: $B: V \to L^2(0,T;H)$ is a bounded linear operator.

Assumption H3.3: $F: H \to H$ is a nonlinear operator for which there exists a positive constant M_F such that

$$\|F(y_1) - F(y_2)\|_H \le M_F \|y_1 - y_2\|_H, \text{ for all } y_1, \ y_2 \in H .$$

Zhou considers an intercept system related to (1.1) on the interval $[t_0, T]$ with a given initial value $y_0 \in H$ prescribed at the time t_0 by

$$\begin{cases} y'(t) = Ay(t) + F(y(t)) + B_{(t_0,T)}u(\cdot)(t), \ 0 < t < T \\ y(t_0) = y_0 \end{cases} \tag{1.2}$$

where the state trajectory $y:[t_0,T] \to H$ belongs to $L^2(t_0,T;H)$, the control $u(\cdot)$ belongs to *V*, and $B_{(t_0,T)}: V \to L^2(t_0,T;H)$ is a bounded linear operator for which

$$B_{(t_0,T)}u(\cdot)(t) = B_{(0,T)}u(\cdot)(t), \text{ for all } t_0 \le t \le T .$$

Assumptions H3.1 – H3.3, taken together, imply that the Cauchy problem (1.2) has a unique mild solution $y(\cdot) \in L^2(t_0,T;H)$ satisfying the variation of parameters formula

$$y(t) = S(t-t_0)y_0 + \int_{t_0}^{t} S(t-s)\left[F(y(s)) + B_{(t_0,T)}u(\cdot)(s)\right]ds, \quad t_0 \le t \le T . \tag{1.3}$$

Motivated by Definition 2.1, we can use (1.3) to formulate the definition of approximate controllability as follows.

Definition 3.1 Let $y_0 \in H$. System (1.2) is *approximately controllable on* $[t_0, T]$ if the reachable set given by

$$R(t_0, z_0) = \left\{ \xi_T \,\middle|\, \exists u \in V \text{ such that } \xi_T = S(T-t_0)y_0 + \int_{t_0}^{T} S(t-s)\left[F(y(s)) + B_{(t_0,T)}u(\cdot)(s)\right]ds \right\} \tag{1.4}$$

where $y(\cdot)$ is the solution of (1.2) corresponding to *u*, is dense in *H*, for every initial condition $y_0 \in H$.

Equivalently, for any $\varepsilon > 0$ and $\xi_T \in H$ there exists a control $u_\varepsilon(\cdot) \in V$ such that

$$\left\| \xi_T - \left(S(T-t_0)y_0 + \int_{t_0}^{T} S(t-s)\left[F(y_\varepsilon(s)) + B_{(t_0,T)}u_\varepsilon(\cdot)(s) \right] ds \right) \right\|_H < \varepsilon , \tag{1.5}$$

where $y_\varepsilon(\cdot)$ is the solution of (1.2) corresponding to $u_\varepsilon(\cdot) \in V$.

The following is the main sufficient condition imposed on (1.1) in order to guarantee the approximate controllability of (1.1) on [0, *T*]:

Assumption H3.4: For every $\varepsilon > 0$ and $p(\cdot) \in L^2\left(t_0, T; H\right)$, there exists $u(\cdot) \in V$ such that

i.) $\left\| \int_{t_0}^{T} S(t-s)\left[p(s) + B_{(t_0,T)}u(\cdot)(s) \right] ds \right\|_H < \varepsilon$;

ii.) there exists a positive constant M_B such that $\left\| B_{(t_0,T)}u(\cdot) \right\|_{L^2(t_0,T;H)} \le M_B \left\| p(\cdot) \right\|_{L^2(t_0,T;H)}$, for all $p(\cdot) \in L^2\left(t_0, T; H\right)$; and

iii.) The product $M_B M_F (T-t_0)$ is sufficiently small.

The following theorem is established in (Zhou, 1983).

Theorem 3.2 System (1.1) is approximately controllable on $[0,T]$ (in the sense of Definition 3.1, suitably modified) if assumptions H3.1 – H3.3 are satisfied and there exists some $t_0 \in \left[0, T\right]$ for which assumption H3.4 holds.

Remark Assumption H3.4 implies that the intercept system (1.2) corresponding to t_0 is approximately controllable. Also, assumption H3.4 is satisfied if, for instance, the range of the operator *B* is dense in $L^2(0,T;H)$.

Naito (Naito, 1987) studied (1.1) (with $y_0 = 0$, for simplicity) under arguably simpler assumptions that avoid imposing inequality conditions involving the system components. Precisely, assumption H3.1 was weakened to

Assumption H3.5: $A : dom(A) \subset H \to H$ is a linear operator that generates a strongly continuous semigroup $\{S(t): 0 \le t \le T\}$ on *H*.

This assumption, together with assumptions H3.2 and H3.3, ensure the uniqueness of mild solutions to (1.1), for every $u \in L^2(0,T;V)$.

The following three conditions were imposed in place of assumption H3.4:

Assumption H3.6: The solution mapping $W : L^2(0,T;V) \to C([0,T];H)$ defined by

$$(Wy)(t) = y(t) = \int_0^t S(t-s)\left[F(y(s)) + (Bu)(s)\right]ds, \quad 0 \le t \le T$$

is compact. (Here, $C([0,T];H)$ is the space of continuous *H*-valued functions on [0,*T*].)

Assumption H3.7: For every $p \in L^2(0,T;H)$, there exists $q \in \mathrm{cl}(\mathrm{rng}(B))$ such that

$$\int_0^T S(T-s)p(s)ds = \int_0^T S(T-s)q(s)ds .$$

Assumption H3.8: $F : H \to H$ is such that there exists a positive constant $\widehat{M_F}$ such that

$$\|F(h)\|_H \le \widehat{M_F}, \quad \text{for all } h \in H .$$

Remarks

1. If assumption H3.5 is strengthened to "*A* generates a *compact* semigroup on *H*," then assumption H3.6 automatically holds.

2. Assumption H3.7 is equivalent to the condition $L^2(0,T;H) = \mathrm{cl}(\mathrm{rng}(B)) + \ker(\Phi)$, where the mapping $\Phi : L^2(0,T;H) \to H$ is defined by $\Phi(p) = \int_0^T S(T-s)p(s)ds$.

The following result is established in (Naito, 1987).

Theorem 3.3 If assumptions H3.2, H3.3, and H3.5 – H3.8 are satisfied, then (1.1) (with $y_0 = 0$) is approximately controllable on $[0,T]$.

An estimate of the diameter of the set of admissible controls is also obtained when the following additional assumption is imposed:

Assumption H3.9: There exists a positive constant $\overline{M_B}$ such that

$$\|u\|_{L^2(0,T;V)} \le \overline{M_B} \|Bu\|_{L^2(0,T;H)} \text{ , for all } u \in L^2(0,T;V) . \tag{1.6}$$

Yamamoto and Park (Yamamoto and Park, 1990) studied the same problem for parabolic equations with uniformly bounded linear part.

In tandem with the 1983 work, a slightly more general abstract semilinear system of the form

$$\begin{cases} y'(t) = Ay(t) + F(y(t),u(t)) + Bu(t), \ 0 < t < T \\ y(0) = y_0 \end{cases} \tag{1.7}$$

where $y:[0,T] \to H$ is the state trajectory and H is a Hilbert space, the control $u(\cdot)$ belongs to $L^2(0,T;U)$ and takes values in a Hilbert space U, $A: H \to H$ is a linear operator on H, $F: H \times U \to H$ is a nonlinear operator, and $B: U \to H$ is a bounded linear operator is considered in (Zhou, 1984).

The following assumptions are imposed in this paper:

Assumption H3.10: $B: V \to H$ is a bounded linear operator such that there exists a positive constant $\overline{M_B}$ for which

$$\|u\|_V \le \overline{M_B} \|Bu\|_H \text{ , for all } u \in U . \tag{1.8}$$

Assumption H3.11: $F: H \times U \to H$ is a nonlinear operator for which there exist positive constants M_1 and M_2 such that

$$\|F(y_1,u_1) - F(y_2,u_2)\|_H \le M_1 \|y_1 - y_2\|_H + M_2 \|u_1 - u_2\|_U , \text{ for all } y_1, \ y_2 \in H \text{ and } u_1, \ u_2 \in U .$$

Assumptions H3.5, H3.10, and H3.11, taken together, imply that for each $u \in L^2\left(0,T;H\right)$ and $y_0 \in H$, the Cauchy problem (1.7) has a unique mild solution $y(\cdot) = y(\cdot\,; y_0, u) \in L^2\left(0,T;H\right)$ satisfying

$$y(t) = S(t-t_0)y_0 + \int_{t_0}^{t} S(t-s)\left[F(y(s),u(s)) + Bu(s)\right]ds, \quad 0 \le t \le T . \tag{1.9}$$

The approximate controllability result for (1.7) is formulated using the family of associated quadratic optimal control problems given by

$$\inf_{\varepsilon>0} J_\varepsilon(u;h), \text{ where } J_\varepsilon(u;h) = \|y(T)-h\|_H^2 + \varepsilon \|u(\cdot)\|_{L^2(0,T;U)}^2 . \tag{1.10}$$

It can be shown that for every $h \in H$ and $\varepsilon > 0$, there exists a control $u_\varepsilon(\cdot) \in L^2\left(0,T;U\right)$ such that

$$J_\varepsilon(u_\varepsilon;h) = \inf_{u \in L^2\left(0,T;U\right)} J_\varepsilon(u;h). \qquad (1.11)$$

The following theorem is established.

Theorem 3.4 System (1.7) is approximately controllable on $[0,T]$ (in the sense of Definition 3.1, suitably modified) if and only if for every $h \in H$,

$$\lim_{\varepsilon \to 0} J_\varepsilon(u_\varepsilon;h) = 0. \qquad (1.12)$$

Next, Mahmudov (Mahmudov, 2003) investigated the following variant of (1.7)

$$\begin{cases} y'(t) = Ay(t) + F(t, y(t), u(t)) + Bu(t), \ \ 0 < t < T \\ y(0) = y_0 \end{cases} \qquad (1.13)$$

where $F:[0,T] \times H \times U \to H$ is a nonlinear operator and all other mappings are as in (1.7).

Approximate controllability results for (1.13), as well as a stochastic variant of (1.13), are formulated under the basic assumption of approximate controllability of the associated linear system. The following assumptions are imposed:

Assumption H3.12: H is a separable reflexive Banach space and U is a Hilbert space.

Assumption H3.13: $A: dom(A) \subset H \to H$ generates a compact semigroup $\left\{S(t): 0 \le t \le T\right\}$ on *H*.

Assumption H3.14: $F:[0,T] \times H \times U \to H$ is a continuous nonlinear operator for which there exists a positive constant $\overline{\overline{M_F}}$ such that

$$\left\|F(t,h,u)\right\|_H \le \overline{\overline{M_F}}, \ \ \text{for all } (t,h,u) \in [0,T] \times H \times U.$$

Assumption H3.15: The linear system corresponding to (1.13) is approximately controllable.

Under these conditions, the following theorem is proved.

Theorem 3.5 If assumptions H3.12 – H3.15 are satisfied, then (1.13) is approximately controllable on $[0,T]$.

The proof of this result involves a constructive approach for approximate controllability of semilinear evolution equations and is based on a characterization of a symmetric positive operator in terms of strong (weak) convergence of a sequence of (resolvent) operators. This result can be applied to both distributed and lumped controls.

Dauer and Mahmudov (Dauer and Mahmudov, 2002) consider a version of (1.13) with finite delay. Precisely, let $C = C\left(\left[-h,0\right];H\right)$ be the space of continuous functions from $[-h,0]$ into H equipped with the usual supremum norm and consider the system

$$\begin{cases} y_t(0) = y(t) = S(t)\phi(0) + \int_{t_0}^{t} S(t-s)\left[F(s,y_s,u(s)) + Bu(s)\right]ds, & 0 \le t \le T \\ y_0(\theta) = \phi(\theta), \quad -h \le \theta \le 0 \end{cases} \tag{1.14}$$

where $\left\{S(t): 0 \le t \le T\right\}$ is a linear semigroup on *H*, $F:[0,T]\times H \times U \to H$ is a nonlinear operator; $\phi \in C$; and *y*, *u*, and *B* are as in (1.13).

The following assumptions are imposed:

Assumption H3.16: $\left\{S(t): 0 \le t \le T\right\}$ is a compact semigroup on *H*.

Assumption H3.17: $F:[0,T]\times C \times U \to H$ is a continuous nonlinear operator for which there exists a positive constant M_F^* such that

$$\left\|F(t,\phi,u)\right\|_H \le M_F^*, \quad \text{for all } (t,\phi,u) \in [0,T]\times C \times U .$$

Assumption H3.18: $\alpha R\left(\alpha, \Phi_0^T\right) \to 0$ as $\alpha \to 0^+$ in the strong operator topology, where $R\left(\alpha,\Phi_0^T\right) = \left(\alpha I + \Phi_0^T\right)^{-1}$ and $\Phi_0^T = \int_0^T S(T-s)BB^*S^*(T-s)ds$.

Remark Assumption H3.18 holds if and only if the associated linear system

$$\begin{cases} y'(t) = Ay(t) + Bu(t), & 0 < t < T \\ y(0) = \phi(0) \end{cases}$$

is approximately controllable on $[0,T]$.

The following theorem is established:

Theorem 3.6 If assumptions H3.16 – H3.18 are satisfied, then (1.14) is approximately controllable on $[0,T]$.

The papers summarized above have served as springboards for related investigations in a plethora of different directions in which the growth conditions imposed on the nonlinear forcing terms are weakened to growth conditions of a non-Lipschitz type and those in which the classical initial condition is replaced by a so-called nonlocal initial condition. Motivated by physical problems, Byszewski (Byszewski, 1991) introduced the notion of a Cauchy problem equipped with a nonlocal initial condition of the form

$$y(0)+g(y)=y_0, \tag{1.15}$$

where $g:C\left(\left[0,T\right]:H\right)\rightarrow H$ is a given function satisfying an appropriate growth condition. The literature regarding existence theory of solutions of abstract evolution equations equipped with such nonlocal initial conditions has since flourished. More recently, researchers have become interested in controllability issues of such systems; see (Mahmudov and Zorlu, 2013), (Sakthivel *et al.*, 2011), (Ge, *et al.*, 2016), and (Zhang, *et al*., 2015).

Approximate controllability results for abstract second-order equations governed by the generator of a strongly continuous cosine family have been established using similar fixed-point techniques (Mahmudov and McKibben, 2013). The extension of the theory to fractional nonlinear differential *inclusions* has been provided in (Sakthivel, *et al.*, 2013) and to the case of time-dependent differential inclusions with finite delay and impulsive effects in (Grudzka and Rykaczewski, 2014).

The extension of the theory of approximate controllability of deterministic abstract systems to abstract stochastic systems driven by Brownian motion, fractional Brownian motion, and Levy jump processes continues to be a very active research direction. We refer the reader to the following papers and the references included therein: (Bashirov and Mahmudov, 1999), (Mahmudov, 2003), and (Mahmudov and McKibben, 2006).

4. Fractional Approximate Controllability

The concept of a non-integral derivative arises frequently in the mathematical modeling of phenomena across disciplines. Indeed, fractional derivatives arise in the mathematical modeling of phenomena in areas such as aerodynamics, chemistry, control theory, electrodynamics of complex media, engineering, physics, and porous media; see the references in (Sakthivel, *et al*., 2011) and (Zhang, *et al.*, 2015). Fractional derivatives can more effectively used to describe memory and the hereditary properties of certain materials and processes. Heymans and Podlubny (Heymans and Podlubny, 2006) showed that fractional derivatives of quantities had actual physical meaning in viscoelasticity. Further, Wang points out in (Wang, *et al.*, 2015) that fractional diffusion equations more accurately model anomalous diffusion in which a plume of particles spreads in a manner different from what the classical diffusion equation predicts. A specific third-order dispersion equation is investigated in (Sakthivel and Mahmudov, 2011).

We begin by considering abstract fractional nonlinear control systems of the form

$$\begin{cases} {}^{c}D_t^q y(t) = Ay(t) + (Bu)(t) + f(t, y(t)), \; 0 < t < T \\ y(0) = y_0 \end{cases} \tag{1.16}$$

where the state function $y:[0,T] \to H$ and H is a Hilbert space; the control function $u(\cdot)$ belongs to $L^2(0,T;V)$ where V is a another Hilbert space V; $A: dom(A) \subset H \to H$ is a linear operator on H that generates a strong continuous semigroup $\{S(t): t > 0\}$ on H; $B: V \to H$ is a bounded linear operator; $f:[0,T] \times H \to H$ is a given function; $y_0 \in H$; and ${}^{c}D^q$ is the Caputo fractional derivative of order $0 < q < 1$ defined as follows:

Definition 4.1

i.) The *fractional integral of order* $\alpha > 0$ with lower limit 0 for a function $g:[0,\infty) \to \mathbb{R}$ is defined as

$$I^{\alpha} g(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{g(s)}{(t-s)^{1-\alpha}} ds, \quad t > 0, \tag{1.17}$$

provided the right-side is pointwise defined on $[0,\infty)$, where Γ is the gamma function.

ii.) The *Riemann-Liouville derivative of order* $\alpha > 0$ with lower limit 0 for a function $g:[0,\infty) \to \mathbb{R}$ is given by

$${}^{L}D^{\alpha} g(t) = \frac{1}{\Gamma(n-\alpha)} \cdot \frac{d^n}{dt^n} \int_0^t \frac{g^{(n)}(s)}{(t-s)^{\alpha+1-n}} ds, \quad t > 0, \; n-1 < \alpha < n.$$

iii.) The *Caputo derivative of order* $\alpha > 0$ with lower limit 0 for a function $g:[0,\infty) \to \mathbb{R}$ is given by

$${}^{c}D^{\alpha} g(t) = {}^{L}D^{\alpha} \left(g(t) - \sum_{k=0}^{n-1} \frac{t^k}{k!} g^{(k)}(0) \right), \quad t > 0, \; n-1 < \alpha < n.$$

The integral equation

$$y(t) = \widehat{\psi_q}(t) y_0 + \int_0^t (t-s)^{q-1} \psi_q(t-s) f(s, y(s)) ds + \int_0^t (t-s)^{q-1} \psi_q(t-s) Bu(s) ds, \tag{1.18}$$

where

$$\widehat{\psi_q}(t)=\int_0^\infty \xi_q(\theta)S\left(t^q\theta\right)d\theta,$$
$$\psi_q(t)=q\int_0^\infty \theta\xi_q(\theta)S\left(t^q\theta\right)d\theta,$$
$$\xi_q(\theta)=\tfrac{1}{q}\theta^{-1-\frac{1}{q}}\overline{w}_q\left(\theta^{-\frac{1}{q}}\right)\geq 0,$$
$$\overline{w}_q(\theta)=\tfrac{1}{\pi}\sum_{n=1}^{\infty}(-1)^{n-1}\theta^{-qn-1}\cdot\tfrac{\Gamma(nq+1)}{n!}\sin(n\pi q),\ \text{ for } \theta>0,$$

is associated with (1.16) in order to define the following notion of mild solution of (1.16) for a given control *u*.

Definition 4.2 A continuous function $y:[0,T]\to H$ is a mild solution of (1.16) if *y* satisfies the integral equation (1.18).

The notion of approximate controllability of the linear fractional control system

$$\begin{cases} D_t^q y(t)=Ay(t)+(Bu)(t), & 0<t<T \\ y(0)=y_0 \end{cases} \tag{1.19}$$

is a natural generalization of the notion of approximate controllability for the linear control system (0.1). The first of two main results established in (Sakthivel, *et al.*, 2011) concerns the approximate controllability of (1.16) and is formulated under the following assumptions:

Assumption H4.1: $A:dom(A)\subset H\to H$ generates a compact semigroup $\{S(t):0\leq t\leq T\}$ on *H*.

Assumption H4.2: The function $f:[0,T]\times H\to H$ is such that

i.) for every $t\in[0,T]$, the function $f(t,\cdot):H\to H$ is continuous;

ii.) for every $h\in H$, the function $f(\cdot,h):[0,T]\to H$ is strongly measurable;

iii.) $f:[0,T]\times H\to H$ is jointly continuous on $[0,T]\times H$ and there exists a positive constant M_f for which $\|f(t,h)\|_H\leq M_f$, for all $(t,h)\in[0,T]\times H$; and

iv.) there exists a constant $q_1\in[0,q]$ and $m\in L^{\frac{1}{q_1}}\left([0,T];(0,\infty)\right)$ such that $\|f(t,h)\|_H\leq m(t)$, for all $h\in H$ and almost all $t\in[0,T]$.

Assumption H4.3: The linear system (1.18) is approximately controllable on $[0,T]$.

Using the technique introduced in (Mahmudov, 2003), Sakthivel et al. establishes the following result:

Theorem 4.3 If assumptions H4.1 – H4.3 are satisfied, then the fractional system (1.16) is approximately controllable on $[0,T]$.

The second main result in this paper concerns a nonlocal variant of (1.16), namely

$$\begin{cases} {}^cD_t^q y(t) = Ay(t) + (Bu)(t) + f(t, y(t)), \ \ 0 < t < T \\ y(0) + g(y) = y_0 \end{cases} \tag{1.20}$$

where $g : C\left(\left[0,T\right] : H\right) \to H$ is a given function satisfying the following global Lipschitz growth condition:

Assumption H4.4: There exists a positive constant M_g such that for all $z_1,\ z_2 \in H$,

$$\left\| g(z_1) - g(z_2) \right\|_{C([0,T]\,;\,H)} \le M_g \left\| z_1 - z_2 \right\|_H .$$

The second main result established in (Sakthivel, *et al.*, 2011) is

Theorem 4.4 If assumptions H4.1 – H4.4 are satisfied, then the nonlocal fractional system (1.19) is approximately controllable on $[0,T]$.

Next, Liu and Li (Liu and Li, 2015) investigate the following fractional control system related to (1.16):

$$\begin{cases} D_t^q y(t) = Ay(t) + (Bu)(t) + f(t, y(t)), \ \ 0 < t < T \\ I_t^{1-q}\, y(t)\big|_{t=0} = y_0 \end{cases} \tag{1.21}$$

Here, *A*, *f*, *B*, and y_0 are as in (Sakthivel, *et al.*, 2011), with the exception that the Hilbert space *V* is specifically taken to be $L^p\left([0,T];U\right)$, *p* > 1, (the space of *p*-integrable *U*-valued functions on (0,*T*)), the initial condition is specified for $I_t^{1-q}\, y(t)\big|_{t=0}$ (cf. (1.17)) instead of *y*(0), and the Caputo fractional derivative ${}^cD_t^q$ in (1.16) is replaced by the Riemann-Liouville fractional derivative D_t^q.

The integral equation associated with (1.21) (similar to (1.18)) is given by

$$y(t)=t^{q-1}\psi_q(t)y_0+\int_0^t (t-s)^{q-1}\psi_q(t-s)f(s,y(s))ds+\int_0^t (t-s)^{q-1}\psi_q(t-s)Bu(s)ds. \quad (1.22)$$

This leads to the following definition of a mild solution of (1.21):

Definition 4.5 A $y:[0,T]\to H$ is a mild solution of (1.21) if *y* satisfies the integral equation (1.22) and belongs to $\mathbb{C}_{1-q}\left([0,T];H\right)=\left\{y:t^{1-q}y(t)\in\mathbb{C}\left([0,T];X\right)\right\}$, which is a Banach space when equipped with the norm $\|y\|_{\mathbb{C}_{1-q}}=\sup\left\{t^{1-q}\|y(t)\|_H : t\in[0,T]\right\}$.

The results established in this paper are motivated by those in (Sakthivel, *et al.*, 2011), but the underlying assumptions imposed on the semigroup generated by *A* is weakened and the growth conditions imposed on the nonlinear term *f* are different. Precisely, the following conditions are imposed:

Assumption H4.5: $A:dom(A)\subset H\to H$ generates a differentiable semigroup $\left\{S(t):0\le t\le T\right\}$ on *H*.

Assumption H4.6: The function $f:[0,T]\times H\to H$ is such that

i.) there exists a function $\phi(\cdot)\in L^p\left([0,T];(0,\infty)\right)$ (where $p>\frac{1}{q}$) and a positive constant M_f such that

$$\|f(t,h)\|_H\le\phi(t)+M_f\, t^{1-q}\|h\|_H \text{, for almost every } t\in[0,T] \text{ and all } h\in H \text{; and}$$

ii.) there exists a positive constant $\overline{M_f}$ such that

$$\|f(t,h_1)-f(t,h_2)\|_H\le\overline{M_f}\, t^{1-q}\|h_1-h_2\|_{\mathbb{C}_{1-q}} \text{, for all } h_1,h_2\in H \text{ and } t\in[0,T].$$

Assumption H4.7: For every $\varepsilon>0$ and $\varphi\in L^p\left(\left[0,T\right];H\right)$, there exists a control $u\in L^p\left(\left[0,T\right];U\right)$ for which

i.) $\left\|\int_0^T (T-s)^{q-1}\psi_q(T-s)\left(\varphi(s)-(Bu)(s)\right)ds\right\|_H<\varepsilon$;

ii.) there exists a positive constant M_B (independent of $\varphi\in L^p\left(\left[0,T\right];H\right)$) such that $\|Bu(\cdot)\|_{L^p}<M_B\|\varphi(\cdot)\|_{L^p}$; and

iii.) $\dfrac{M_S \overline{M}_f M_B}{\Gamma(q)} \cdot \left(\dfrac{p-1}{pq-1}\right)^{\frac{p-1}{p}} T^{1-\frac{1}{p}} E_q\left(M_S \overline{M}_f T\right) < 1$, where $M_S = \sup\left\{\|S(t)\|_H : 0 \le t \le T\right\}$.

It can be shown that under assumptions H4.5 and H4.6, for each control function $u \in V$, the control system (1.21) has a unique mild solution in the sense of Definition 4.5.

Under these assumptions, Liu and Li prove the following theorem:

Theorem 4.6 If assumptions H4.5 – H4.7 are satisfied, then the fractional system (1.21) is approximately controllable on $[0,T]$.

Wang, et al. (Wang, *et al.*, 2015) investigate the related fractional *partial* differential system

$$\begin{cases} {}^{c}D_t^q y(z,t) = Ay(z,t) + (Bu)(z,t) + f(t, y(z,t)), \quad 0 \le z \le 2\pi, \ 0 < t < T \\ \quad y(z,0) = 0 \end{cases} \tag{1.23}$$

where the state function $y(\cdot,t)$ and the control function $u(\cdot,t)$ take values in $L^2\left(0,2\pi\right)$; $A : dom(A) \subset L^2\left(0,2\pi\right) \to L^2\left(0,2\pi\right)$ is a linear operator; and $B : L^2\left(0,2\pi\right) \to L^2\left(0,2\pi\right)$ is a bounded linear operator.

Definition 4.7 A function $y(z,t) \in L^2\left(\left[0,T\right]; L^2\left(0,2\pi\right)\right)$ is a mild solution of (1.23) if *y* satisfies the integral equation

$$y(z,t) = \int_0^t (t-s)^{q-1} \psi_q(t-s)\left(Bu(z,s) + f(s, y(z,s))\right) ds,$$

for all $0 \le z \le 2\pi$ and $0 < t < T$.

They establish the approximate controllability of (1.23) under the following conditions:

Assumption H4.8: $A : dom(A) \subset L^2\left(0,2\pi\right) \to L^2\left(0,2\pi\right)$ generates a compact analytic semigroup $\left\{S(t) : 0 \le t \le T\right\}$ on $L^2\left(0,2\pi\right)$;

Assumption H4.9: The function $f : [0,T] \times L^2\left(0,2\pi\right) \to L^2\left(0,2\pi\right)$ is such that

i.) for every $t \in [0,T]$, the function $f\left(t,\cdot\right) : L^2\left(0,2\pi\right) \to L^2\left(0,2\pi\right)$ is continuous;

ii.) there exists a function $\eta \in L^{\frac{1}{j}}\left([0,T]; L^2\left(0,2\pi\right)\right)$, where $0 < j < q$, such that

$$\| f(t,h_1) - f(t,h_2)\|_{L^2(0,2\pi)} \le \eta(t)\|h_1 - h_2\|_{L^2(0,2\pi)},$$

for all $(t,h_1),\ (t,h_2) \in [0,T]\times L^2\left(0,2\pi\right)$; and

iii.) there exists a positive constant $\overline{\overline{M_f}}$ such that $\| f(t,h)\|_{L^2(0,2\pi)} \le \overline{\overline{M_f}}$, for all $(t,h) \in [0,T]\times L^2\left(0,2\pi\right)$.

Assumption H4.10: $\lambda R\left(\lambda, \Phi_0^T\right) \to 0$ as $\lambda \to 0^+$ in the strong operator topology. (Here,

$$\Phi_0^T = \int_0^T (T-s)^{q-1}\psi_q(T-s)BB^*T\psi_q^*(T-s)ds \text{ and } R\left(\lambda,\Phi_0^T\right) = \left(\lambda I + \Phi_0^T\right)^{-1},$$

where B^* and ψ_q^* are the adjoints of the operators B and ψ_q, respectively.

Assumption H4.11: $\dfrac{M_S\, q}{\Gamma(1+q)}\|\eta\|_{L^{1/j}}\left(\dfrac{1-j}{q-j}\right)^{1-j} T^{q-1} < 1$, where $M_S = \sup\left\{\|S(t)\|_H : 0 \le t \le T\right\}$.

Under these assumptions, Wang, et al. prove the following theorem:

Theorem 4.8 If assumptions H4.8 – H4.11 are satisfied, then the fractional partial differential system (1.23) is approximately controllable on $[0,T]$.

The theory of control systems with impulses has been studied with increasing vigor over the past decade. Impulse control problems arise, for instance, when one considers models of chaotic systems and investment decisions in financial mathematics. Sudden jumps at certain time points in the evolution of processes arising in biotechnology, pharmacokinetics, population dynamics, and radiation of electromagnetic waves, for instance, are examples of such impulsive behavior.

The first study on impulsive systems that we consider is the work in (Ge, *et al.*, 2016). Ge, et al. consider a nonlocal variant of (1.16) equipped with jumps at finitely many time points. Specifically, they study

$$\begin{cases} {}^cD_t^q y(t) = Ay(t) + (Bu)(t) + f(t, y(t)),\ \ 0 < t < T,\ t \ne t_i \\ y(0) = g(y), \\ \Delta y\big|_{t \ne t_i^+} = J_i\left(y(t_i)\right),\ i = 1,\dots, p \end{cases} \tag{1.24}$$

where $0 < t_1 < t_2 < ... < t_p < T$ and $\Delta y\big|_{t \neq t_i^+} = y(t_i^+) - y(t_i^-),$ where $y(t_i^+)$ and $y(t_i^-)$ represent the left- and right-limits of $y(t)$ at $t = t_i$. The operators A and B are as above and the functions f, g, and J_i are suitable mappings on which conditions will be imposed later in the manuscript.

The collection of functions given by

$$PC\left([0,T];H\right) = \left\{ y:[0,T] \to H \;\middle|\; \begin{array}{l} y(t) \text{ is continuous at } t \neq t_i, \text{ left continuous at } t_i \\ \text{and } \lim\limits_{t \to t_i^+} y(t) \text{ exists for } i = 1,2,...,p \end{array} \right\} \quad (1.25)$$

equipped with the norm $\|y\|_{PC} = \sup\left\{\|y(t)\|_H : 0 \le t \le T\right\}$ is a Banach space. Also, for any $R > 0$, the following set is needed in the formulation of Ge et al.'s main results:

$$\omega_R = \left\{ y \in PC\left([0,T];H\right) : y(t) \in B_R, \text{ for all } t \in [0,T] \right\}, \quad (1.26)$$

where $B_R = \left\{ h \in H : \|h\|_H \le R \right\}$ is the closed ball of radius R in the space H.

The work in (Ge, *et al.*, 2016) seems to have been inspired by the final remark in (Sakthivel, *et al.*, 2011). In addition to assumptions H4.1 and H4.2, they impose the following restriction on the jump functions J_i :

Assumption H4.12: The functions $J_i : H \to H \; \left(i = 1,..., p\right)$ are continuous.

The authors establish two main approximate controllability results for (1.24) depending on the growth condition imposed on the nonlocal function g. For the first result, they assume:

Assumption H4.13: The mapping $g : PC\left([0,T];H\right) \to H$ is such that $g(0) = 0$ and there exists a positive constant M_g for which

$$\|g(h_1) - g(h_2)\|_H \le M_g \|h_1 - h_2\|_{PC} \text{ , for all } h_1, h_2 \in H .$$

Under assumptions H4.1, H4.2 (suitably modified so that condition (iii) is applied in the second variable in $f(t, h)$ to the closed ball B_R , for each $R > 0$, rather than on the entire space), H4.12, and H4.13, the authors use a standard compactness argument to show that for each bounded control $u \in L^2(0,T;V)$, the fractional impulsive system (1.24) has at least one mild solution on $[0,T]$ provided that the data are sufficiently small.

Remarks

1. Uniqueness is no longer guaranteed because of the weakened growth restriction on *f*.

2. By the phrase "the data are sufficiently small," we simply mean that an inequality is imposed where a (typically rather technical-looking) quantity involving the growth constants and parameters arising in the Cauchy problem is assumed to be less than 1. Imposing such a condition is standard practice, is usually done so in order to ensure that fixed-point methods can successfully be used to establish the result.

In order to obtain the approximate controllability result for (1.24), the authors further assume that assumption H4.10 holds, as well as the following enhancement of assumptions H4.12 and H4.13:

Assumption H4.14: The functions $g : PC\left([0,T];H\right) \to H$ and $J_i : H \to H \; \left(i = 1,\ldots, p\right)$ are uniformly bounded on $PC\left([0,T];H\right)$.

The following result is then established:

Theorem 4.9 If assumptions H4.1, H4.2 (suitably modified) and H4.12 – H4.14 are satisfied, then (1.24) is approximately controllable on $[0,T]$, provided that the data are sufficiently small.

The second main result in (Ge, *et al.*, 2016) is obtained by weakening the growth condition on the nonlocal initial condition *g*. Specifically, assumption H4.13 is weakened to the following:

Assumption H4.15: The function $g : PC\left([0,T];H\right) \to H$ is continuous and for any $R \geq 0$ and any $x, y \in w_R$, there exists $\delta = \delta(R) \in (0,t_1)$ such that

$$x(s) = y(s), \text{ for all } s \in \left[\delta, T\right] \Rightarrow g(x) = g(y).$$

As before, under this assumption, provided that the data are sufficiently small, the authors are able to establish the existence of a mild solution of (1.24), and subsequently the approximate controllability of (1.24) under the additional assumption H4.14 using a similar compactness argument.

Next, Zhang, et al. study the following nonlocal impulsive fractional system similar to (1.24):

$$\begin{cases} {}^cD_t^q y(t) = Ay(t) + (Bu)(t) + f(t, y(t), Gy(t)), \; 0 < t < T, \; t \neq t_i \\ y(0) = g(y) + y_0, \\ \Delta y(t_i) = J_i\left(y(t_i^-)\right), \; i = 1,\ldots, p \end{cases} \tag{1.27}$$

Here, $A: dom(A) \subset H \to H$ is a linear operator that generates an *analytic* semigroup on *H*. As such, the space *H* used in the domains and ranges of the mappings *B*, *f*, *g*, etc, is often replaced by the domain of the fractional power of *A*, $dom\left(A^{\alpha}\right)$ $(0 < \alpha < 1)$, which is known to be a Banach space when equipped with the graph norm of A^{α}, denoted by $\|\cdot\|_{\alpha}$. More specifically, $B: V \to dom\left(A^{\alpha}\right)$ is a bounded linear operator; $J_i : dom\left(A^{\alpha}\right) \to dom\left(A^{\alpha}\right)$ $(i = 1, \ldots, p)$, $f:[0,T] \times dom\left(A^{\alpha}\right) \times dom\left(A^{\alpha}\right) \to H$, and $g: PC\left([0,T]; dom\left(A^{\alpha}\right)\right) \to dom\left(A^{\alpha}\right)$.

The mapping *G* defined by

$$Gy(t) = \int_0^t K(t,s)y(s)ds \tag{1.28}$$

is a Volterra integral operator with kernel $K \in \mathbb{C}\left(\Delta; [0,\infty)\right)$, where $\Delta = \left\{(t,s): 0 \le s \le t \le T\right\}$.

The space in (1.25) is naturally modified for studying (1.27) by replacing *H* by $dom\left(A^{\alpha}\right)$; we denote this space by $PC\left([0,T]; dom\left(A^{\alpha}\right)\right)$. A mild solution of (1.27) is then naturally defined as follows:

Definition 4.10 A function $y \in PC\left([0,T]; dom\left(A^{\alpha}\right)\right)$ is a mild solution of (1.27) if for any $u \in L^2(0,T;V)$, the integral equation

$$\begin{aligned} y(t) = \widehat{\psi_q}(t)\left(y_0 + g(y)\right) + \int_0^t (t-s)^{q-1}\psi_q(t-s)\left(Bu(s) + f(s, y(s), Gy(s))\right)ds \\ + \sum_{0<t_i<t} \widehat{\psi_q}(t-t_i)J_i\left(y(t_i^-)\right) \end{aligned} \tag{1.29}$$

is satisfied.

The authors use arguments similar to those in the papers reviewed above (this time using the Krasnoselskii fixed-point theorem) to argue that (1.27) is approximately controllable on $[0,T]$ under the following assumptions:

Assumption H4.16: The function $f:[0,T] \times dom\left(A^{\alpha}\right) \times dom\left(A^{\alpha}\right) \to H$ is continuous and satisfies the following:

i.) there exist a constant $\gamma \in (0, (1-\alpha)q)$ and functions $\varphi_R \in L^{\frac{1}{\gamma}}\left([0,T]; (0,\infty)\right)$ such that

$$\sup\left\{\|f(t,y,Gy)\|_H : \|y\|_\alpha \le R\right\} \le \varphi_R(t) \text{ and } \varliminf_{r\to\infty} \frac{\|\varphi_R\|_{L^{1/\gamma}}}{R} = \sigma < \infty \text{; and}$$

ii.) f is bounded in $dom\left(A^\alpha\right)$.

Assumption H.17: The mapping $g : PC\left([0,T]; dom\left(A^\alpha\right)\right) \to dom\left(A^\alpha\right)$ satisfies the usual global Lipschitz condition (cf. assumption H4.13, for instance) and g is bounded in $dom\left(A^\alpha\right)$.

Assumption H.18: The mappings $J_i : dom\left(A^\alpha\right) \to dom\left(A^\alpha\right)$ $(i = 1,..., p)$ satisfy the usual global Lipschitz condition and are bounded in $dom\left(A^\alpha\right)$.

Assumption H.19: The linear system associated with (1.27) is approximately controllable on $[0,T]$.

Precisely, the authors prove the following result:

Theorem 4.11 If assumptions H4.16 - H4.19 are satisfied, then (1.27) is approximately controllable on $[0,T]$, provided that the data is sufficiently small.

We wrap up our discussion of fractional control systems with a brief comment on Riemann-Liouville fractional differential *inclusions*. The focus of this discussion is the recent work of Yang and Wang (Yang and Wang, 2016). They investigate the approximate controllability for Riemann-Liouville neutral fractional differential inclusions of the form

$$\begin{cases} {}^L D_t^q \left[y(t) - h(t, y(t))\right] \in Ay(t) + (Bu)(t) + F(t, y(t)), \ \ 0 < t < T \\ I_t^{1-q}\left[y(t) - h(t, y(t))\right]\Big|_{t=0} = y_0 \end{cases} \tag{1.30}$$

where $A : dom(A) \subset H \to H$ is a linear operator that generates an analytic semigroup on *H*, *B* is a bounded linear operator (as above), $h : [0,T] \times H \to H$ is a given mapping, and $F : [0,T] \times H \to \wp(H) = \left\{Y \in 2^Y : Y \neq \varnothing\right\}$ is a nonempty, bounded, closed, convex multifunction.

Definition 4.12 A function $y \in \mathbb{C}_{1-\alpha}\left(\left[0,T\right]; H\right)$ is a mild solution of (1.30) if the following are satisfied:

i.) $I_t^{1-q}\left[y(t) - h(t, y(t))\right]\Big|_{t=0} = y_0$, ii.) there exists $f \in \left\{ f \in L^1\left(\left[0,T\right];H\right): f(t) \in F(t, y(t)), \text{ for a.e. } t \in [0,T] \right\}$ for which

$$y(t) = t^{\alpha-1}\psi_\alpha(t)y_0 + h(t, y(t)) + \int_0^t (t-s)^{\alpha-1}\psi_q(t-s)f(s)ds$$
$$+\int_0^t (t-s)^{\alpha-1}A\psi_\alpha(t-s)h(s, y(s))ds + \int_0^t (t-s)^{\alpha-1}\psi_\alpha(t-s)Bu(s)ds, \ t \in [0,T]$$

The existence of a mild solution of (1.30), as well as the approximate controllability of (1.30) on [0,*T*], are obtained under conditions very similar to those used in (Sakthivel, *et al.*, 2011) and (Zhang, *et al.*, 2015) with the exception that particular care must now be taken to suitably modify the conditions previously imposed on the single-valued nonlinearity *f* to obtain appropriate conditions for the multifunction *F*. Otherwise, the arguments employed are similar in spirit to those used in the papers above.

The study of the approximate controllability of fractional differential systems continues to thrive. Other works concerning the approximate controllability of fractional integrodifferential systems (Ganesh, *et al.*, 2013), delay fractional systems (Kumar and Sukavanam, 2013), and other nonlinear fractional differential systems as in (Mahmudov and Zorlu, 2014) and (Sakthivel, *et al.*, 2013) have appeared in the literature recently. Also, Mahmudov and McKibben studied such abstract systems in which the derivative of the state function was a generalized version of the Riemann-Liouville derivative in (Mahmudov and McKibben, 2015). There are many other directions that can be taken with this work that will have even broader applications in different disciplines.

5. Concluding Remarks

We have reviewed the approximate controllability of infinite-dimensional linear, nonlinear, and fractional control systems in this paper. The assumption of the approximate controllability of the linear part associated with nonlinear systems was key, natural, and readily verifiable, and permeated investigations of approximate controllability for second-order systems, impulsive systems, delay systems, fractional differential systems, and stochastic systems. The theory discussed has applications in a great many different disciplines and offers a unifying structure in which to study such phenomena. There is much work that lies ahead.

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