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Jonathan Eskeldson
University of Oregon

Miriam Jaffe
Johns Hopkins University

Viorel Nitica
West Chester University of Pennsylvania, vnitica@wcupa.edu

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A metric on max-min algebra

Jonathan Eskeldson, Miriam Jaffe, and Viorel Nitica

ABSTRACT. Using the characterization of the segments in the max-min semimodule $\mathcal{B}^n$, provided by Nitica and Singer in *Contributions to max-min convex geometry. I: Segments*. Linear Algebra and its Applications **428**, (2008), 1439–1459, we find a class of metrics on the $\mathcal{B}^n$. One of them is given by the Euclidean length of the max-min segment connecting two points. The max-min segments are complicated and consist of several Euclidean segments pointing in a finite number of fixed directions. The number of directions increases with the dimension of the semimodule. Each metric in our class is associated with a weighting function, for which we give some characterization. None of these metrics is a quasiconvex metric. Nevertheless, a somehow weaker condition always holds.

1. Introduction

Consider the set $\mathcal{B} = [0, 1]$ endowed with the operations $\oplus = \max, \otimes = \min$. This is a distributive lattice known as *boolean algebra* or *fuzzy algebra* and it can be considered as a semiring equipped with addition max and multiplication min. The identity for the addition is 0 and the identity for the multiplication is 1. Both operations are idempotent, $\max(a, a) = a$ and $\min(a, a) = a$, and closely related to the order:

$$\max(a, b) = b \Leftrightarrow a \leq b \Leftrightarrow \min(a, b) = a. \tag{1.1}$$

For standard literature on lattices and semirings see e.g. [**2**] and [**5**].

We consider $\mathcal{B}^n$, the cartesian product of n copies of $\mathcal{B}$, and equip this cartesian product with the operations of taking componentwise addition:

$$x \oplus y := (\max(x_1, y_1), \max(x_2, y_2), \ldots, \max(x_n, y_n))$$

for $x = (x_1, x_2, \ldots, x_n), y = (y_1, y_2, \ldots, y_n) \in \mathcal{B}^n$, and scalar multiplication:

$$a \otimes x := (\min(a, x_1), \min(a, x_2), \ldots, \min(a, x_n))$$

for $a \in \mathcal{B}$, $x = (x_1, x_2, \ldots, x_n) \in \mathcal{B}^n$. Thus $\mathcal{B}^n$ becomes a semimodule over $\mathcal{B}$ [**5**].

One can canonically introduce a convex structure on $\mathcal{B}^n$.

DEFINITION 1.1. *A subset C of $\mathcal{B}^n$ is said to be* max-min convex *if the relations*

$$x, y \in C, \alpha, \beta \in \mathcal{B}, \alpha \oplus \beta = 1$$

imply

$$(\alpha \otimes x) \oplus (\beta \otimes y) \in C.$$

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The interest in max-min convexity is motivated by the study of tropically convex sets, analogously defined over the max-plus semiring $\mathbb{R}_{\max}$, which is the completed set of real numbers $\mathbb{R} \cup \{-\infty\}$ endowed with operations of idempotent addition $\max(a, b)$ and multiplication $a + b$. Introduced in [**14, 15**], tropical convexity and its lattice-theoretic generalizations received much attention and rapidly developed over the last decades. For a basic textbook on the subject see [**6**]. Another relevant reference is the book [**1**].

The results in max-min convexity are many times parallel to those in max-plus convexity, with different proofs, but some noticeable differences were observed. For example, separation of two convex sets by hyperplanes is not always possible in max-min convexity [**7**]. Several other papers investigating max-min convexity that appeared in the last years are [**8**], [**9**], [**11**], [**12**], [**13**]. A recent survey of this subject, containing also new material such as max-min counterparts of Carathéodory, Radon and Helly theorems, is [**10**].

The main goal of this paper is to introduce a metric on $\mathcal{B}^n$. The metric we derive is closely related to the structure of max-min segments as presented in [**11**]. The analytic properties of this metric will be further investigated in the future.

In the usual linear space $\mathbb{R}^n$, a line segment is defined as the set of all convex combinations of the endpoints:

$$[x, y] = \{tx + sy | 0 \leq t, s \text{ and } t + s = 1\}. \tag{1.2}$$

We note that in the above definition, 1 is the multiplicative identity and 0 is the additive identity. By analogy, this gives the following definition of a max-min segment:

Definition 1.2. *The max-min segment joining $x, y \in \mathcal{B}^n$ is defined by the following equation:*

$$[x, y] = \{(\alpha \otimes x) \oplus (\beta \otimes y) \,|\alpha \oplus \beta = 1\}. \tag{1.3}$$

Remark 1.3. *a) If $x = (x_1, ..., x_n), y = (y_1, ..., y_n) \in \mathcal{B}^n$,* (1.3) *is equivalent to*

$$[x, y] = \left\{ \left(\max(\min(\alpha, x_1), \min(\beta, y_1)), ..., \max(\min(\alpha, x_n), \min(\beta, y_n)) \right) \middle| \max(\alpha, \beta) = 1 \right\}.$$

b) Definition 1.1 simply says that a set is convex if together with any two points, contains the full max-min segment joining the points.

Recall the partial ordering on $\mathcal{B}^n$. If $x = (x_1, ..., x_n), y = (y_1, ..., y_n) \in \mathcal{B}^n$, then $x \leq y$ if and only if $x_i \leq y_i$ for all $1 \leq i \leq n$. In this case, we call the pair (x, y) and the segment $[x, y]$ *commensurable.* If $x \not\leq y$ and $y \not\leq x$, we call the pair (x, y) and the segment $[x, y]$ *incommensurable.*

It is showed in [**11**] that max-min segments are composed of concatenations of so called *elementary segments.* Elementary segments are usual Euclidean segments in $\mathcal{B}^n$ that keep certain coordinates fixed, and change the values of the rest of the coordinates uniformly from a to b, for some $a, b \in \mathcal{B}$. A parametrization of an elementary segment is given by

$$\begin{cases} x_{i_1} = c_1, \ldots, x_{i_k} = c_k, \\ x_{i_{k+1}} = \cdots = x_{i_n} = t, t \in [a, b], \end{cases}$$

or

$$\begin{cases} x_{i_1} = c_1, \ldots, x_{i_k} = c_k, \\ x_{i_{k+1}} = \cdots = x_{i_n} = a + b - t, t \in [a, b], \end{cases}$$

where $0 \leq k \leq n$, $a < b, a, b, c_1, \ldots, c_k \in \mathcal{B}$ are constants, $x_{i_1}, \ldots, x_{i_k}$ are the coordinates kept fixed and $x_{i_{k+1}}, \ldots, x_{i_n}$ are the variable coordinates. For example, the Euclidean segment between $(0, 0, 0, .5, 1)$ and $(1, 1, 1, .5, 1)$ is an elementary segment with the coordinates x_4, x_5 kept fixed and x_1, x_2, x_3 variable coordinates.

Given $x, y \in \mathcal{B}^n$, [**11**] also presented an algorithmic method for constructing the max-min segment between them. If $x \leq y$, this method proceeds by starting at x, and then increasing the least coordinate of x until it reaches the value of some other coordinate of x or y. If we reach another coordinate of x, we start increasing both coordinates simultaneously; if we reach the same coordinate of y, then we stop increasing this x coordinate. We continue this process increasing multiple x coordinates until a y coordinate is reached, in which case we stop increasing the corresponding x coordinate and continue increasing the rest. This process stops when the point y is reached. For the example of max-min segment shown in Figure 1, connecting the commensurable points (x_1, x_2, x_3) and (y_1, y_2, y_3), the elementary segments are:

$$\begin{aligned}
S_1 &= \{(t, x_2, x_3) | t \in [z_1, z_2]\},\\
S_2 &= \{(t, t, x_3) | t \in [z_2, z_3]\},\\
S_3 &= \{(t, t, t) | t \in [z_3, z_4]\},\\
S_4 &= \{(t, t, y_3) | t \in [z_4, z_5]\},\\
S_5 &= \{(y_1, t, y_3) | t \in [z_5, z_6]\}.
\end{aligned}$$

If $x \not\leq y$ and $y \not\leq x$, then the max-min segment from x to y is a concatenation of segments from x to $\max(x, y)$, and from $\max(x, y)$ to y, which reduces the segment $[x, y]$ to the concatenation of two commensurable segments. For the example of max-min segment shown in Figure 2, connecting the incommensurable points $(x_1, x_2, x_3, x_4, x_5)$ and $(y_1, y_2, y_3, y_4, y_5)$, the elementary segments are:

$$\begin{aligned}
S_1 &= \{(t, x_2, x_3, x_4, x_5) | t \in [z_1, z_2]\},\\
S_2 &= \{(t, x_2, t, x_4, x_5) | t \in [z_2, z_4]\},\\
S_3 &= \{(t, x_2, y_3, x_4, x_5) | t \in [z_4, z_7]\},\\
S_4 &= \{(y_1, x_2, y_3, t, x_5) | t \in [z_7, z_{10}]\},\\
S_5 &= \{(y_1, x_2, y_3, y_4, z_6 + z_9 - t) | t \in [z_6, z_9]\},\\
S_6 &= \{(y_1, z_5 + z_6 - t, y_3, y_4, z_5 + z_6 - t) | t \in [z_5, z_6]\},\\
S_7 &= \{(y_1, z_3 + z_5 - t, y_3, y_4, y_5) | t \in [z_3, z_5]\}.
\end{aligned}$$

We mention that, using ideas similar to those in this paper, a metric on the max-plus semimodule is introduced in [**4**].

The rest of the paper is organized as follows. In Section 2 we define a distance on $\mathcal{B}^n$ by taking the Euclidean length of the max-min segment connecting two points. This distance is shown to be a metric. In Section 3 we show that if we weight the elementary segments differently, this procedure still produces a metric. Moreover, sufficient and necessary conditions for a general weight to produce a metric are given. Finally, in Section 4, we will show that no weighted metric is quasi-convex. Nevertheless, a somehow weaker condition always holds.

2. A Metric in $\mathcal{B}^n$

In this section, our goal is to introduce a metric on $\mathcal{B}^n$.

Definition 2.1. *We call two elementary segments $[x_1, y_1]$ and $[x_2, y_2]$ in $\mathcal{B}^n$* adjacent *if $y_1 = x_2$.*

Definition 2.2. *Given $x, y \in \mathcal{B}^n$, we call a finite sequence of adjacent elementary segments a* path from x to y *if the first segment in the sequence begins at x and the last segment in the sequence ends at y.*

Note that any max-min segment is a path and that there are paths that are not max-min segments. The sequence of elementary segments belonging to a path is ordered.

Definition 2.3. *We say that an elementary segment in a path* starts *at the point where it adjoins the previous elementary segment, and* terminates *at the point where it adjoins the next elementary segment. We say that an elementary segment* increases a coordinate *if the value of the coordinate at the initial point is less than the value of the coordinate at the end point.*

Definition 2.4. *An elementary segment that changes p coordinates is called a* p-sector.

For example, an elementary segment that changes the first two coordinates is a 2-sector. Note that any elementary segment is a p-sector for some p.

Definition 2.5. *The* length of a p-sector *with the variable coordinates between a and b is defined to be its Euclidean length, that is $\sqrt{p}(b - a)$. The* length of a path *is equal to the sum of the lengths of all elementary segments belonging to the path.*

REMARK 2.6. *The notion of length introduced above defines on $\mathcal{B}^n$ a structure of length space as presented, for example, in* [**3**]. *In particular, the metric we introduce here on $\mathcal{B}^n$ is an* intrinsic metric. *We refer to Chapter 2 of* [**3**] *for details.*

The following elementary lemma is needed in the future.

LEMMA 2.7. *Let $p \geq 2$ integer and $p_1, p_2, \ldots, p_k, k \geq 2$, strictly positive integers such that $p_1 + p_2 + \cdots + p_k = p$. Then:*

$$\sqrt{p} < \sqrt{p_1} + \sqrt{p_2} + \cdots + \sqrt{p_k}. \tag{2.1}$$

PROOF. Square both sides of (2.1). □

A path is a way to get from x to y in $\mathcal{B}^n$. To prove triangle inequality for the length introduced above, we show that the shortest path between any two points is given by the max-min segment joining them.

First we show that we can disregard many paths that clearly are not the shortest; for example, paths that retrace their steps. This will limit the number of paths we have to consider, and make them more well behaved.

LEMMA 2.8. *Suppose a path from x to y has two elementary segments in $\mathcal{B}^n$ such that one increases a coordinate, and the other decreases the same coordinate. Then we can find a shorter path from x to y by projecting a portion of the path on an Euclidean plane parallel to one of the coordinate planes. Moreover, after a finite number of such transformations, the resulting path does not have any pair of elementary segments that increase and decrease the same coordinate.*

PROOF. Assume a path from x to y has two elementary segments such that one increases a coordinate, and the other decreases the same coordinate. Without loss of generality, let this coordinate be the x_1 coordinate, and suppose that the first decreasing segment comes after an increasing segment. We may have several elementary segments in between for which the x_1 coordinate is constant.

Let c_1 be the value of the x_1-coordinate where the x_1-coordinate stops increasing. Then there is an elementary segment in the path whose x_1-coordinate increases, starting at $c_1 - \epsilon_1$ and ending at c_1, and there exists an elementary segment in the path whose x_1-coordinate decreases, starting at c_1 and ending at $c_1 - \epsilon_2$, with ϵ_1 and ϵ_2 positive. Take c to be $c_1 - \min\{\epsilon_1, \epsilon_2\}$.

Consider the hyperplane defined by $x_1 = c$. We can pick two points in the path belonging to this hyperplane so that the portion of the path between them does not lie inside the hyperplane. We project the portion of the path between these two points onto this hyperplane. Then we observe that this projection is still a path, and that it is a shorter path.

We prove that the projection of an elementary segment is an elementary segment, and that the projections of two adjacent elementary segments are adjacent.

An elementary segment fixes some coordinates, and changes the values of the rest. For each elementary segment, the first coordinate will project to c, and the rest of the coordinates will remain as they were before. So the projection is still an elementary segment. Now, consider two adjacent elementary segments. Define $P = (p_1, ..., p_n)$ to be the point where the first terminates and the second one begins. When we project the first elementary segment on the hyperplane, its terminal point projects to $(c, p_2, ..., p_n)$, and when we project the second elementary segment, its initial point projects to $(c, p_2, ..., p_n)$, thus their projections are adjacent. Therefore the projection of the portion of the path onto this hyperplane is still a path.

To show that the projection is shorter, suppose the elementary segment has $k \leq n$ variable coordinates and $n-k$ fixed coordinates, where the variable coordinates go from some a to some b. Then the length of the elementary segment is $\sqrt{k}(b-a)$. The length of the projection is either $\sqrt{k}(b-a)$ when the x_1-coordinate is fixed, or $\sqrt{k-1}(b-a)$ when the x_1-coordinate is variable. Thus, the length of the projection is less than the length of the elementary segment, so the length of the projected path must be less than the length of the path. As we have at least one elementary segment that changes the x_1-coordinate, the length of the projected path is actually strictly less then the length of the initial path.

Now, consider the original path from x to y, and replace the portion that we projected with its projection. Then this is a shorter path from x to y.

To show now that only a finite number of such transformations are necessary in order to obtain a path that does not have any pair of elementary segments that increase and decrease the same coordinate, observe that due to our choice of the constant c the projection reduces the number of elementary segments for which the x_1 coordinate is not constant by 1. □

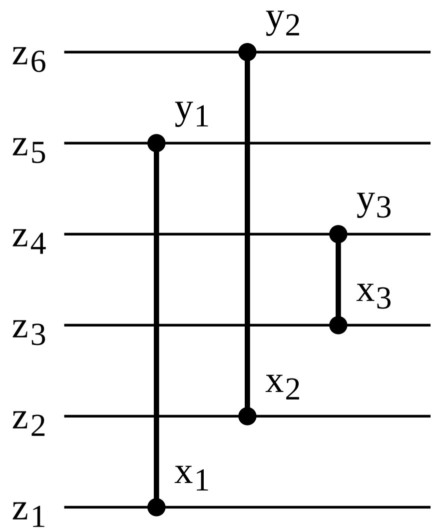

FIGURE 1. An example of how to view a path between commensurable points in $\mathbb{R}^3$.

Intuitively, the next corollary states that if a path from x to y goes outside of the straight box with x and y at opposite corners, then there is a shorter path joining x and y.

COROLLARY 2.9. *Let $x, y \in \mathcal{B}^n$. Then for all coordinates i, the shortest path from x to y does not include points with a value in the i-th coordinate either less than $\min(x_i, y_i)$ or greater than $\max(x_i, y_i)$.*

PROOF. If a path's i-th coordinate exceeds $\max(x_i, y_i)$ for some i, the i-th coordinate has to increase to exceed the maximum coordinate value and then needs to decrease to reach the terminal point. After applying Lemma 2.8, we can find a shorter path. A similar argument holds for decreasing the i-th coordinate to a value less than $\min(x_i, y_i)$. □

We use these results to show that the shortest path between x and y is given by the max-min segment which connects them.

THEOREM 2.10. *The path with the shortest length between x and y in $\mathcal{B}^n$ has length equal to the length of the max-min segment $[x, y]$.*

PROOF. We first assume that $x \leq y$.

Let $x = (x_1, x_2, ..., x_n), y = (y_1, y_2, ..., y_n) \in \mathcal{B}^n$. Write the coordinate values $x_1, ..., x_n,\ y_1, ..., y_n$ in increasing order and relabel them as $z_1 \leq z_2 \leq \cdots \leq z_{2n}$. We divide $\mathcal{B}$ into $2n + 1$ intervals: $[0, z_1]$, $[z_1, z_2]$, $[z_2, z_3], \ldots, [z_{2n-1}, z_{2n}], [z_{2n}, 1]$. See Figure 1 and Figure 2. We note that some of these intervals may consist of a single point and that each one of them gives a parametrization of a p-sector in P. The number p associated to the interval $[z_i, z_{i+1}]$ is exactly the number of coordinates $j, 1 \leq j \leq n$, for which $x_j \leq z_i \leq z_{i+1} \leq y_j$ and the length of the p-sector is $\sqrt{p}(z_{i+1} - z_i)$.

Consider now an arbitrary path P joining x and y. Due to Lemma 2.8 and the fact that $x \leq y$, we can assume that the path does not decrease any coordinate. Consider an elementary segment I belonging to the path which increases some coordinates from α to β. Then by Corollary 2.9 necessarily

$$z_1 \leq \alpha < \beta \leq z_{2n}.$$

Moreover, if the coordinate j increases, then

$$x_j \leq \alpha < \beta \leq y_j.$$

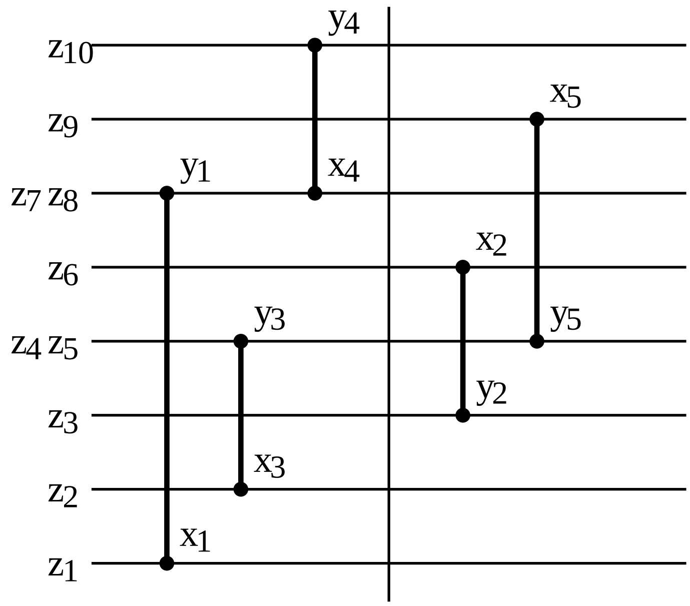

FIGURE 2. An example of how to view a path between incommensurable points in $\mathbb{R}^5$. The increasing coordinates are 1, 3, and 4, and the decreasing coordinates are 2 and 5.

We divide the interval $[\alpha, \beta]$ in subintervals, completely contained in the intervals $[z_i, z_{i+1}]$. This divides the elementary segment I, and the whole path P, into a finite set of elementary segments for which the increasing parameter stays inside one of the intervals $[z_i, z_{i+1}]$. Some of these segments may overlap. After one more subdivision we may assume that the intervals defining the increasing parameter for the elementary segments in P either coincide, or have no more then a point of intersection.

Let J_1, J_2 be two such elementary segments for which the increasing parameter stays in the interval $[\gamma, \delta]$. The coordinates j that can increase in either J_1 or J_2 are exactly those for which $x_j \leq \gamma \leq \delta \leq y_j$. Moreover, as no coordinate can increase and then decrease, the sets of coordinates increasing in the segment J_1 and the set of coordinates increasing in J_2 are disjoint. As each coordinate j has to be covered by the path P from x_j to y_j, we see that the elementary segments in P with the increasing parameter in the interval $[\gamma, \delta]$ determine a partition of the coordinates j for which $x_j \leq \gamma \leq \delta \leq y_j$.

Assume that there are p such coordinates. The length contributed by the max-min segment for the value of the increasing parameter in the interval $[\gamma, \delta]$ is $\sqrt{p}(\delta - \gamma)$. We compute now the length contribution in the path P. Assume that there are k elementary segments $J_1, J_2, \ldots, J_k$ in P for which the increasing parameter belongs to the interval $[\gamma, \delta]$. Assume that the partition of the set of increasing coordinates has cardinalities $p_1, p_2, \ldots, p_k$. So p_i are strictly positive integers with sum p. The sum of the lengths of the intervals $J_1, J_2, \ldots, J_k$ is

$$(\sqrt{p_1} + \sqrt{p_2} + \cdots + \sqrt{p_k})\,(\delta - \gamma).$$

It follows now from Lemma 2.7 that:

$$\sqrt{p}(\delta - \gamma) \leq (\sqrt{p_1} + \sqrt{p_2} + \cdots + \sqrt{p_k})\,(\delta - \gamma),$$

thus the increment in length of the max-min segment due to the interval $[\gamma, \delta]$ is less or equal then the increment in length in the path P due to the interval $[\gamma, \delta]$. As this happens for each interval $[\gamma, \delta]$, we conclude that the max-min segment reaches the minimum length.

Now consider the incommensurable case, when $x \not\leq y$ and $y \not\leq x$.

Let P be the shortest path connecting x and y. We can partition the set of coordinates in 3 subsets: one subset consists of coordinates that are constant for any elementary segment in P, one subset contains the coordinates i such that $x_i < y_i$, which we call "positive" coordinates, and the last subset contains the coordinates j such that $x_j > y_j$, which we call "negative" coordinates. Indeed, if P uses an elementary segment that affects both positive and negative coordinates, it either has to increase both or decrease both. However, because the positive coordinates must increase and the negative coordinates must decrease, one coordinate must both increase and decrease. Then by Lemma 2.8 there is a shorter path. Therefore, the shortest path cannot use such segments. Thus we can partition the elementary segments that the shortest path uses into those that affect only positive coordinates and those that change only negative coordinates.

The length of P is the sum of the Euclidean lengths of the elementary segments that use the positive coordinates, respectively negative coordinates. Because two elementary segments, one increasing the coordinates and one decreasing them, change different coordinates, we can change the order of the elementary segments in P without affecting the length of P. Consider a new path P' consisting of two subpaths: the first one starts at x and is made of all elementary segments in P that increase some of the positive coordinates, listed in the order in which they appear in P, until they reaches the point $\max(x, y)$; the second subpath starts at $\max(x, y)$ and is made of all elementary segments in P that decrease some of the negative coordinates, listed in the order in which they appear in P, until they reaches the point y. The problem has now been reduced to the path P', which has the same length with P, and consists of two subpaths joining commensurable points, one from x to $\max(x, y)$, and one from $\max(x, y)$ to y. Each of these subpaths is commensurable, so we can use the case studied before. □

We will prove now that d is a metric.

Corollary 2.11. *Let $x, y \in \mathcal{B}^n$, and let $d(x, y)$ be the Euclidean length of the max-min segment connecting them. Then d is a metric.*

Proof. Obviously the segment from x to y is the same as the segment from y to x. Thus $d(x, y) = d(y, x)$. Also $d(x, y) = 0$ if and only if $x = y$.

Finally, let $x, y, z \in \mathcal{B}^n$, and consider a path from x to z that goes through y. By Theorem 2.10, the length of this path is greater than or equal to the length of the max-min segment connecting x and z, so $d(x, y) + d(y, z) \geq d(x, z)$. Thus d is a metric. □

3. Weighted Metrics in $\mathcal{B}^n$

In this section we generalize Theorem 2.10 by using other methods of weighting elementary segments. Previously, the weight of an elementary segment that changed k coordinates was $\sqrt{k}$. Now we even allow to weight differently elementary segments changing the same number of coordinates. For example, the weight of an elementary segment changing only the first coordinate can be different from the weight of an elementary segment changing only the second. Clearly, the weights must be positive and finite.

Definition 3.1. *The* length of a p-sector *with the variable coordinates between a and b that has assigned the weight w is defined to be $w(b-a)$. The* length of a path *is equal to the sum of the lengths of all elementary segments belonging to the path. If $x, y \in \mathcal{B}^n$, define $d(x, y)$ to be the length of the max-min segment from x to y.*

Theorem 3.2. *Let $\emptyset \neq S \subseteq \{1, ..., n\}$. Let $w_S > 0$ be the weight of the elementary segment changing only the coordinates contained in S and let $w_\emptyset = 0$. Then the length d introduced above is a metric if the following conditions hold:*

$$w_A + w_B \geq w_{A-(A\cap B)} + w_{B-(A\cap B)}, A, B \subseteq \{1, ..., n\} \tag{3.1}$$

$$w_{S_1} + w_{S_2} + \cdots + w_{S_k} \geq w_{S_1\cup\cdots\cup S_k}, S_1, ..., S_k \subseteq \{1, ..., n\} \text{ disjoint sets.} \tag{3.2}$$

Proof. The proof is similar to that of Theorem 2.10.

We show first a modified version of Lemma 2.8. Suppose that a path increases and then decreases the same coordinate, say x_1. Let c_1 be maximum of x_1 before it decreases. Let L^+ be the last elementary segment that increases the x_1 coordinate before this point, and let L^- be the first elementary segment that decreases it after this point. There is a maximal $\epsilon > 0$ for which $(c_1 - \epsilon, c_1)$ is included in the parameter interval of both L^+, L^-. Define A to be the set of coordinates changed by L^+, and B to be the set of coordinates changed by L^-.

Consider the portion of the path between L^+ and L^-. First, assume that there are no elementary segments between L^+ and L^- that change any coordinate in both A and B.

We project the portion of the path between L^+ and L^- (inclusively) onto the subspace defined by $x_i = c_1 - \epsilon$ for all $i \in A \cap B$. An elementary segment that does not change any coordinate in both A and B, is projected to an elementary segment of equal length. The only elementary segments which change any coordinate in $A \cap B$ are L^+ and L^- by assumption. The portions of L^+ and L^- that are projected both have the parameter intervals of length ϵ. By (3.1) we find that $\epsilon \cdot w_A + \epsilon \cdot w_B \geq \epsilon \cdot w_{A-(A\cap B)} + \epsilon \cdot w_{B-(A\cap B)}$. Since $\epsilon \cdot w_{A-(A\cap B)} + \epsilon \cdot w_{B-(A\cap B)}$ is the weighted length of the projection of L^+ and L^-, and $\epsilon \cdot w_A + \epsilon \cdot w_B$ is the weighted length of L^+ and L^-, and the length of each intermediate segment remains the same on the projection, the length of the projection is smaller. If we construct a new path in which we use the projection between L^+ and L^- inclusively, then the new path is shorter than the original.

Now assume that there are segments between L^+ and L^- that change coordinates in both A and B. Choose any such coordinate. Find the value at which it first decreases; call it c_2. By a similar process as before, we can find L_2^+, L_2^-, $\epsilon_2 > 0$, A_2, and B_2, with similar properties as before. If the portion of the path between L_2^+ and L_2^- doesn't contain any segments changing coordinates in $A_2 \cap B_2$, then as before, we can find a shorter path. Otherwise, continue this process until we find some c_k, L_k^+, L_k^-, $\epsilon_k > 0$, A_k, and B_k such that no portion of the path between L_k^+ and L_k^- changes any of the coordinates in $A_k \cap B_k$. This process will terminate, because there are a finite number of elementary segments between L^+ and L^-. Once we've reached this step, there is a shorter path. Thus if a path increases and decreases a coordinate, a shorter path exists.

Next, we proceed as we did in the the proof of Theorem 2.10. First, we assume that $x \leq y$.

For a given path P joining x, y, we take a partition of P in elementary segments for which the parameter belongs to a subinterval $[\gamma, \delta]$ inside a subinterval $[z_i, z_{i+1}]$, with z_i defined as in the proof of Theorem 2.10. As before, the elementary segments in P with the increasing parameter in the interval $[\gamma, \delta]$ determine a partition of the coordinates j for which $x_j \leq \gamma \leq \delta \leq y_j$. Denote the set of such coordinates by S. The length contributed by the weighted max-min segment for the value of the parameter in $[\gamma, \delta]$ is $w_s(\delta - \gamma)$. We compute the length contribution of the path P for the parameter value in $[\gamma, \delta]$. Let the partition of S be $S = S_1 \cup S_2 \cup \cdots \cup S_k$, which determine the elementary segments $J_1, J_2, \ldots, J_k$ in P. The lengths of $J_1, J_2, \ldots, J_k$ are $w_{S_1}(\delta - \gamma), w_{S_2}(\delta - \gamma), \ldots, w_{S_k}(\delta - \gamma)$. Now it follows from (3.2) that the contribution of P is larger then the contribution of the regular max-min segment, and we are done.

As in Theorem 3.4, the incommensurable case reduces to the commensurable one. Consequently d is a metric. □

REMARK 3.3. *The notion of length introduced above defines on $\mathcal{B}^n$ a structure of length space as presented, for example, in* [**3**]. *In particular, the metrics introduced here on $\mathcal{B}^n$ are all* intrinsic metrics. *We refer to Chapter 2 of* [**3**] *for details.*

Next, we show that simple uniform weights produce a metric.

THEOREM 3.4. *Let $x, y \in \mathcal{B}^n$. Let $w_i > 0, 1 \leq i \leq n$ be the weight assigned to an elementary segment in $\mathcal{B}^n$ that changes exactly i coordinates and let $w_0 = 0$. Define $d(x, y)$ to be the sum of the weighted lengths of the elementary segments comprising the max-min segment from x to y. Then d is a metric if for all $1 \leq i, j \leq k \leq n, i + j = k$, the following conditions hold:*

$$w_i + w_j \geq w_k \tag{3.3}$$

$$w_i \leq w_k. \tag{3.4}$$

PROOF. The theorem is a corollary of Theorem 3.2, as (3.4) implies (3.1) and (3.3) implies (3.2). □

THEOREM 3.5. *The conditions* (3.1) *and* (3.2) *for the weights described by Theorem 3.2 are necessary in order for the length of the max-min segment $[x, y], x, y \in \mathcal{B}^n$, to define a metric on $\mathcal{B}^n$.*

Proof. We proceed by showing that whenever one of the conditions (3.1) and (3.2) is removed, the triangle inequality does not hold for all points in $\mathcal{B}^n$. Suppose that (3.1) does not hold for all $A, B \subseteq \{1, ..., n\}$, and define $d(x, y)$ to be the weighted length of $[x, y]$. Then there exists A and B such that

$$w_A + w_B < w_{A-(A\cap B)} + w_{B-(A\cap B)}. \tag{3.5}$$

Now define $x, y, z \in \mathcal{B}^n$ as follows:

$$x_i = \begin{cases} 1, & i \in B - (A \cap B) \\ 0, & \text{otherwise} \end{cases} \qquad y_i = \begin{cases} 1, & i \in A - (A \cap B) \\ 0 & \text{otherwise} \end{cases} \qquad z_i = \begin{cases} 1, & i \in A \cup B \\ 0 & \text{otherwise.} \end{cases}$$

Then the following equalities hold:

$$d(x, y) = w_{A-(A\cap B)} + w_{B-(A\cap B)}, \qquad d(x, z) = w_A, \qquad d(y, z) = w_B. \tag{3.6}$$

Therefore, by (3.5)

$$d(x, z) + d(y, z) = w_A + w_B < w_{A-(A\cap B)} + w_{B-(A\cap B)} = d(x, y) \tag{3.7}$$

which violates the triangle inequality, so d cannot be a metric.

Now, suppose that (3.2) does not hold and the weighting still produces a metric d. There exists some $k \geq 2$ such that for some $S_1, ..., S_k \subseteq \{1, ..., n\}$, with $S_i \cap S_j = \emptyset$ for $i \neq j$, the following is true:

$$w_{S_1} + \cdots + w_{S_k} < w_{S_1\cup\cdots\cup S_k}$$

Now take the points $x = (0, 0, \ldots, 0)$ and y in $\mathcal{B}^n$ defined by

$$y_i = \begin{cases} 1 & \text{if } i \in S_1 \cup \cdots \cup S_k \\ 0 & \text{otherwise} \end{cases}$$

Then $d(x, y) = w_{S_1\cup\cdots\cup S_k}$.

Now, consider the path from x to y that first uses the elementary segment that changes exactly the coordinates in S_1 from 0 to 1. Call z_1 the point at which this path terminates. Now, change all coordinates of z_1 in S_2 from 0 to one, and call the point that this terminates at z_2. This elementary segment is well defined, because all the coordinates in S_2 don't appear in S_1 because $S_1 \cap S_2 = \emptyset$, so they are 0 at z_1. Continue this process until we change all coordinates in z_k from 0 to 1. The path then terminates at y. The length of this path is

$$d(x, z_1) + d(z_1, z_2) + \cdots + d(z_{k-1}, z_k) + d(z_k, y) = w_{S_1} + \cdots + w_{S_k}.$$

By our initial assumption, we find that

$$d(x, z_1) + d(z_1, z_2) + \cdots + d(z_{k-1}, z_k) + d(z_k, y) = w_{S_1} + \cdots + w_{S_k} < w_{S_1\cup\cdots\cup S_k} = d(x, y). \tag{3.8}$$

However, since d is a metric, it follows by repeated application of triangle inequality that

$$d(x, y) \leq d(x, z_1) + d(z_1, z_2) + \cdots + d(z_{k-1}, z_k) + d(z_k, y). \tag{3.9}$$

Combining (3.8) and (3.9) we arrive at the conclusion that

$$d(x, y) \leq d(x, z_1) + d(z_1, z_2) + \cdots + d(z_{k-1}, z_k) + d(z_k, y) < d(x, y).$$

which is a contradiction. Thus our assumption that the weighting produces a metric is false, so d is not a metric.

In conclusion, any relaxing of the conditions in Theorem 3.2 will never yield a metric.

□

4. Lack of Quasiconvexity

A desired property of a metric is quasiconvexity. Quasiconvexity models the well known property from Euclidean geometry that in a triangle with vertices a, b, and c, the distance from a to any point on the segment $[b, c]$ is at most the maximum of the distance between a and b and the distance between a and c. Recall some formal definitions.

DEFINITION 4.1. *The subset S of the metric space (X, d) is called a* metric segment with endpoints $u, v \in X$ *if there exists an isometry $\phi(p) : [0, d(u, v)] \to X$ such that $\phi(0) = u, \phi(d(u, v)) = v$ and $\phi([0, d(u, v)]) = S$.*

The following lemma is immediate.

LEMMA 4.2. *Max-min segments are metric segments for the max-min metrics on $\mathcal{B}^n$ introduced in Sections 2 and 3.*

DEFINITION 4.3. *Let (X, d) a metric space with the metric segment $[a, b]$ defined for all $a, b \in X$. Then d is said to be* quasiconvex *if for all $a, b, c \in X$ and for all $z \in [a, b]$ the following holds:*

$$d(c, z) \le \max(d(c, a), d(c, b)). \tag{4.1}$$

It is well known that the usual Euclidean metric induced on $\mathcal{B}^n$ from $\mathbb{R}^n$ is quasiconvex. However, for $n \ge 2$, it turns out that none of the weighted metrics described in Theorem 3.2 for $\mathcal{B}^n$ are quasiconvex.

THEOREM 4.4. *No weighting of the max-min elementary segments results in a quasiconvex metric for $\mathcal{B}^n$ for $n \ge 2$.*

PROOF. Let w_1 be the weight of the elementary segment that only changes the x_1 coordinate, and let w_2 be the weight of the elementary segment that only changes the x_2 coordinate. Set $b = (0, 100, 0, 0, ..., 0)$, $a = \left(-\frac{1}{w_1}, 100 + \frac{1}{w_2}, 0, 0, ..., 0\right)$, $z = \left(0, 100 + \frac{1}{w_2}, 0, 0, ..., 0\right)$, and $c = \left(-\frac{1}{w_1}, 100, 0, 0, ..., 0\right)$. Then we see that

$$d(a, c) = d(b, c) = 1 \tag{4.2}$$

but

$$d(c, z) = 2 > 1 = \max(d(a, c), d(b, c)) \tag{4.3}$$

Thus for any weighting, we can find points so that the quasiconvexity condition does not hold. □

A somehow weaker condition then quasiconvexity holds.

THEOREM 4.5. *All weightings of the max-min elementary segments produce metrics d such that for all c and for all $z \in [a, b]$ the following holds for all a, b:*

$$d(c, z) \le 2 \max(d(c, a), d(c, b)). \tag{4.4}$$

In addition, 2 is the lowest constant for which this holds for all points in a given weighting.

PROOF. The proof proceeds by repeated application of the triangle inequality. For any three points a, b, and c, and point z on $[a, b]$, we have the following:

$$d(c, z) \le d(a, c) + d(a, z) \tag{4.5}$$

$$d(c, z) \le d(b, c) + d(b, z). \tag{4.6}$$

Summing those two inequalities yields

$$\begin{aligned} 2d(c, z) &\le d(a, c) + d(b, c) + d(a, z) + d(b, z) \\ &\le d(a, c) + d(b, c) + d(a, b) \\ &\le d(a, c) + d(b, c) + d(a, c) + d(b, c). \end{aligned} \tag{4.7}$$

where second line in (4.7) is true because $z \in [a, b]$ implies $d(a, z) + d(z, b) = d(a, b)$.

From (4.7) we have

$$d(c,z) \leq 2\max(d(c,a), d(c,b)). \tag{4.8}$$

The equation holds for all possible weighings that produce a metric.

Now, we prove that 2 is the lowest constant for which this holds for all points. Given an arbitrary weighting, by what was shown in the proof of Theorem 4.4, we can find points a, b, c, z such that $d(c,z) = 2$, but $\max(d(c,a), d(c,b)) = \max(1,1) = 1$, so $d(c,z) = 2\max(d(c,a), d(c,b)$. □

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Department of Mathematics, University of Oregon, Eugene, OR 97403
E-mail address: jeskels@uoregan.edu

Department of Mathematics, John Hopkins University, 3400 North Charles Street, Baltimore, MD 21218
E-mail address: msea0j@gmail.com

Department of Mathematics, West Chester University, West Chester, PA 19380
E-mail address: vnitica@wcupa.edu