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On a class of backward McKean-Vlasov stochastic equations in Hilbert space: Existence and convergence properties

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Abstract: In this paper a new result on the existence and uniqueness of the adapted solution to a backward stochastic evolution equation in Hilbert spaces under non Lipschitz condition is established. The applicability of this result is then illustrated in a discussion of some concrete backward stochastic partial differential equation. Furthermore, stochastic maximum principle for optimal control problems of stochastic systems governed by backward stochastic evolution equations in Hilbert spaces is obtained.

On Backward Stochastic Evolution Equations in Hilbert Spaces and Optimal Control

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Abstract

In this paper a new result on the existence and uniqueness of the adapted solution to a backward stochastic evolution equation in Hilbert spaces under non Lipschitz condition is established. The applicability of this result is then illustrated in a discussion of some concrete backward stochastic partial differential equation. Furthermore, stochastic maximum principle for optimal control problems of stochastic systems governed by backward stochastic evolution equations in Hilbert spaces is obtained.

Key words: Stochastic evolution systems, semilinear systems, maximum principle

1 Introduction

Backward stochastic differential equations (BSDEs for short) have important applications in stochastic control and financial markets. Since the publication of the work of Pardoux and Peng ([13]), many papers have been dedicated to the study of backward stochastic differential equations. Several of these papers (see [7], [8], [9], [18], [17]) have been devoted to the case of BSDE in infinite dimensional spaces. Hu and Peng [7], [8] have considered two cases of semi-linear backward stochastic evolution equations (BSEEs): in the first case the existence of a so-called “mild solution” was established, and in the second case semi-linear backward stochastic partial differential equations (BSPDEs)

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were considered. This kind of equation appears, for example, in the theory of optimal control and controllability for stochastic partial differential equations (see [2]). Maximum principles for stochastic control systems in infinite dimensional spaces were studied by Bensoussan [1], [2], Mahmudov [4], Hu and Peng [8]. Maximum principles for backward stochastic equations in finite dimensional space were studied by Dokuchaev and Zhou [5]. In the present paper, we first establish a result concerning the existence and uniqueness of a mild solution for a class of BSEs with non-Lipschitzian coefficients in Hilbert space that generalizes some of the results in [7], [18], [8]. Secondly, we formulate a stochastic maximum principle for optimal control problems of stochastic systems governed by BSEs in Hilbert spaces and solve a backward linear quadratic stochastic control.

2 Preliminaries

In this section we introduce notations needed to establish our results.

$(\Omega, \mathfrak{F}_T, \mathbf{P})$ is a probability space together with a normal filtration $\{\mathfrak{F}_t, 0 \leq t \leq T\}$, X, U and E are three separable Hilbert spaces. W is a Q -Wiener process on $(\Omega, \mathfrak{F}_T, \mathbf{P})$ with the linear bounded covariance operator such that $\text{tr}Q < \infty$. We assume that there exists a complete orthonormal system $\{e_k\}$ in E , a bounded sequence of nonnegative real numbers λ_k such that $Qe_k = \lambda_k e_k$, $k = 1, 2, \dots$ and a sequence $\{\beta_k\}$ of independent Brownian motions such that

$$\langle w(t), e \rangle = \sum_{k=1}^{\infty} \sqrt{\lambda_k} \langle e_k, e \rangle \beta_k(t), \quad e \in E, \quad t \in [0, T],$$

where $\langle \cdot, \cdot \rangle$ is the inner product in E . Moreover we assume that \mathfrak{F}_t is generated by $w(t)$. Let $L_2^0 = L_2(Q^{1/2}E, X)$ be the space of all Hilbert-Schmidt operators from $Q^{1/2}E$ to X with the inner product $\langle \Psi, \Phi \rangle_{L_2^0} = \text{tr}[\Psi Q \Phi^*]$. $L^2(\Omega, \mathfrak{F}_T, X)$ is the Hilbert space of all \mathfrak{F}_T -measurable square integrable variables with values in a Hilbert space X . $L_{\mathfrak{F}}^2([0, T], X)$ is the Hilbert space of all square integrable and \mathfrak{F}_t -adapted processes with values in X . We recall that f is said to be \mathfrak{F}_t -adapted if $f(t, \cdot) : \Omega \rightarrow X$ is \mathfrak{F}_t -measurable, a.e. $t \in [0, T]$.

For any $\beta \in \mathbb{R}$, define $M_{\beta}[t, T]$ to be the Banach space

$$M_{\beta}[t, T] = L_{\mathfrak{F}}^2(\Omega, C([t, T], X)) \times L_{\mathfrak{F}}^2([t, T], L_2^0)$$

equipped with the norm

$$\|(y, z)\|_{\beta, t}^2 = \mathbf{E} \sup_{t \leq s \leq T} e^{2\beta s} \|y(s)\|^2 + \mathbf{E} \int_t^T e^{2\beta s} \|z(s)\|^2 ds.$$

Since $0 < T < \infty$, all the norms $\|\cdot\|_{\beta,t}$ with different $\beta \in \mathbb{R}$ are equivalent. $M[0, T] = M_0[0, T]$ is the Banach space endowed with the norm

$$\|(y, z)\|^2 = \mathbf{E} \sup_{0 \leq s \leq T} \|y(s)\|^2 + \mathbf{E} \int_0^T \|z(s)\|^2 ds.$$

3 Backward Stochastic Evolution Equations

In this section we study the existence and uniqueness of solution to the following class of backward stochastic evolution equations in a Hilbert space X

$$\begin{cases} dy(t) = -[Ay(t) + F(t, y(t), z(t))] dt - [G(t, y(t)) + z(t)] dw(t), \\ y(T) = \xi, \end{cases} \quad (1)$$

where $A : D(A) \subset X \rightarrow X$ is a linear operator which generates a C_0 -semigroup $\{S(t), 0 \leq t \leq T\}$ on X , $F : [0, T] \times X \times L_2^0 \rightarrow X$ and $G : [0, T] \times X \rightarrow L_2^0$ are given measurable mappings, and $\xi \in L^2(\Omega, \mathfrak{F}_T, X)$.

Definition 1 *A pair of adapted processes $(y, z) \in L_{\mathfrak{F}}^2(\Omega, C([0, T], X)) \times L_{\mathfrak{F}}^2(\Omega \times [0, T], L_2^0)$ is a mild solution of (1) if for all $t \in [0, T]$ they satisfy*

$$\begin{aligned} y(t) &= S(T-t)\xi + \int_t^T S(s-t)F(s, y(s), z(s)) ds \\ &+ \int_t^T S(s-t)[G(s, y(s)) + z(s)] dw(s), \quad P\text{-a.s.} \end{aligned}$$

3.1 Lipschitz case

In this subsection we study existence and uniqueness of mild solution to the equation

$$\begin{aligned} y(t) &= S(T-t)\xi + \int_t^T S(s-t)F(s, y(s), z(s)) ds \\ &+ \int_t^T S(s-t)[g(s) + z(s)] dw(s), \quad P\text{-a.s.} \end{aligned} \quad (2)$$

We make the following assumptions on the function $F : [0, T] \times X \times L_2^0 \rightarrow X$.

(L1) There exists an $L > 0$ such that

$$\|F(t, y, z) - F(t, \bar{y}, \bar{z})\| \leq L(\|y - \bar{y}\| + \|z - \bar{z}\|),$$

for all $t \in [0, T]$, $y, \bar{y} \in X$, $z, \bar{z} \in L_2^0$.

(L2) $F(\cdot, 0, 0) \in L^2([0, T], X)$.

Lemma 2 For any $(f, g) \in L_{\mathfrak{F}}^2([0, T], X) \times L_{\mathfrak{F}}^2([0, T], L_2^0)$ the equation

$$\begin{aligned} y(t) &= S(T-t)\xi + \int_t^T S(s-t)f(s)ds \\ &+ \int_t^T S(s-t)[g(s) + z(s)]dw(s), \quad P\text{-a.s.} \end{aligned} \quad (3)$$

has a unique solution in $M_\beta[0, T]$, and moreover

$$\begin{aligned} &\mathbf{E} \sup_{0 \leq s \leq t} e^{2\beta s} \|y(s)\|^2 + \mathbf{E} \int_t^T e^{2\beta s} \|z(s)\|^2 ds \\ &\leq 24M_S^2 \left(e^{2\beta T} \mathbf{E} \|\xi\|^2 + \frac{1}{2\beta} \int_t^T e^{2\beta r} \mathbf{E} \|f(r)\|^2 dr \right) \\ &+ 2\mathbf{E} \int_t^T e^{2\beta r} \|g(r)\|^2 dr, \end{aligned} \quad (4)$$

where $M_S = \sup \{\|S(t)\|_{\mathfrak{B}(X)}, 0 \leq t \leq T\}$ and $\mathfrak{B}(X)$ is the space of bounded, linear operators on X .

Proof. Equation (3) is a linear BSEE. As such, by Lemma 2.1 [7], it admits a unique mild solution $(y, z) \in M_\beta[0, T]$ given by

$$y(t) = S(T-t) \mathbf{E} \{\xi \mid \mathfrak{F}_t\} + \int_t^T S(s-t) \mathbf{E} \{f(s) \mid \mathfrak{F}_t\} ds, \quad (5)$$

$$\tilde{z}(t) = S(T-t)L(t) - \int_t^T S(s-t)K(t,s)ds, \quad z(t) = \tilde{z}(t) - g(t), \quad (6)$$

where, by the martingale representation theorem (see [7], [4]) the processes $L \in L_{\mathfrak{F}}^2([0, T], L_2^0)$ and $K \in L_{\mathfrak{F}}^2([0, T] \times [0, T], L_2^0)$ satisfy the following relations

$$\begin{aligned} \mathbf{E} \{\xi \mid \mathfrak{F}_t\} &= \mathbf{E}\xi + \int_0^t L(\theta) dw(\theta), \\ \mathbf{E} \{f(s) \mid \mathfrak{F}_t\} &= \mathbf{E}f(s) + \int_0^t K(s, \theta) dw(\theta). \end{aligned}$$

Now, we estimate the solution (y, z) given by (5)-(6) in $M_\beta[t, T]$ for $\beta > 0$. From (5) it follows that

$$\begin{aligned} &\mathbf{E} \sup_{t \leq s \leq T} e^{2\beta s} \|y(s)\|^2 \leq 2M_S^2 \mathbf{E} \sup_{t \leq s \leq T} e^{2\beta s} \|\mathbf{E} \{\xi \mid \mathfrak{F}_s\}\|^2 \\ &+ 2M_S^2 \mathbf{E} \sup_{t \leq s \leq T} e^{2\beta s} \left(\int_s^T \mathbf{E} \{\|f(r)\| \mid \mathfrak{F}_s\} dr \right)^2 = I_1 + I_2. \end{aligned} \quad (7)$$

Standard computations yield:

$$\begin{aligned}
I_2 &\leq 2M_S^2 \mathbf{E} \sup_{t \leq s \leq T} \left(\mathbf{E} \left\{ e^{\beta s} \int_s^T \|f(r)\| dr \mid \mathfrak{F}_s \right\} \right)^2 \\
&\leq 2M_S^2 \mathbf{E} \sup_{t \leq s \leq T} \left(\mathbf{E} \left\{ \sup_{t \leq \tau \leq T} e^{\beta \tau} \int_\tau^T \|f(r)\| dr \mid \mathfrak{F}_s \right\} \right)^2 \\
&\leq 8M_S^2 \mathbf{E} \left(\sup_{t \leq \tau \leq T} e^{\beta \tau} \int_\tau^T \|f(r)\| dr \right)^2 \\
&\leq 8M_S^2 \mathbf{E} \sup_{t \leq \tau \leq T} e^{2\beta \tau} \int_\tau^T e^{-2\beta r} dr \int_\tau^T e^{2\beta r} \|f(r)\|^2 dr \\
&\leq 8M_S^2 \mathbf{E} \sup_{t \leq \tau \leq T} e^{2\beta \tau} \frac{1}{2\beta} [e^{-2\beta \tau} - e^{-2\beta T}] \int_\tau^T e^{2\beta r} \|f(r)\|^2 dr \\
&\leq 8M_S^2 \frac{1}{2\beta} \int_t^T e^{2\beta r} \mathbf{E} \|f(r)\|^2 dr.
\end{aligned}$$

Consequently, by (7)

$$\mathbf{E} \sup_{t \leq s \leq T} e^{2\beta s} \|y(s)\|^2 \leq 8M_S^2 e^{2\beta T} \mathbf{E} \|\xi\|^2 + 8M_S^2 \frac{1}{2\beta} \int_t^T e^{2\beta r} \mathbf{E} \|f(r)\|^2 dr. \quad (8)$$

Next, we estimate z . We have

$$\|\tilde{z}(s)\|^2 \leq 2M_S^2 \|L(s)\|^2 + 2M_S^2 \frac{e^{-2\beta s}}{2\beta} \int_s^T e^{2\beta \theta} \|K(\theta, s)\|^2 d\theta.$$

From here it follows that

$$\begin{aligned}
\mathbf{E} \int_t^T e^{2\beta s} \|\tilde{z}(s)\|^2 ds &\leq 2M_S^2 \mathbf{E} \int_t^T e^{2\beta s} \|L(s)\|^2 ds \\
&\quad + 2M_S^2 \frac{1}{2\beta} \mathbf{E} \int_t^T \int_s^T e^{2\beta \theta} \|K(\theta, s)\|^2 d\theta ds \\
&\leq 8M_S^2 e^{2\beta T} \mathbf{E} \|\xi\|^2 + 2M_S^2 \frac{1}{2\beta} \mathbf{E} \int_t^T \int_0^\theta e^{2\beta \theta} \|K(\theta, s)\|^2 ds d\theta \\
&\leq 8M_S^2 e^{2\beta T} \mathbf{E} \|\xi\|^2 + 4M_S^2 \frac{1}{\beta} \int_t^T e^{2\beta \theta} \mathbf{E} \|f(\theta)\|^2 d\theta. \quad (9)
\end{aligned}$$

The inequalities (8) and (9), together, imply that

$$\begin{aligned}
&\mathbf{E} \sup_{t \leq s \leq T} e^{2\beta s} \|y(s)\|^2 + 2\mathbf{E} \int_t^T e^{2\beta s} \|\tilde{z}(s)\|^2 ds \\
&\leq 24M_S^2 \left(e^{2\beta T} \mathbf{E} \|\xi\|^2 + \frac{1}{2\beta} \int_t^T e^{2\beta r} \mathbf{E} \|f(r)\|^2 dr \right),
\end{aligned}$$

which in turn implies (4). ■

Lemma 3 For any $(f, g) \in L^2_{\mathfrak{F}}([0, T], X) \times L^2_{\mathfrak{F}}([0, T], L^0_2)$, the associated mild solution of (3) satisfies the following estimate

$$\begin{aligned} & \mathbf{E} \sup_{0 \leq s \leq t} \|y(s)\|^2 + \mathbf{E} \int_t^T \|z(s)\|^2 ds \\ & \leq 24M_S^2 \left(\mathbf{E} \|\xi\|^2 + (T-t) \int_t^T \mathbf{E} \|f(r)\|^2 dr \right) + 2\mathbf{E} \int_t^T \|g(r)\|^2 dr. \end{aligned} \quad (10)$$

Proof. See [7]. ■

Theorem 4 BSEE (2) admits a unique solution $(y, z) \in M_\beta[0, T]$.

Proof. For any fixed $(\bar{y}, \bar{z}) \in M_\beta[0, T]$, it follows from (L2) that

$$f(\cdot) = F(\cdot, \bar{y}(\cdot), \bar{z}(\cdot)) \in L^2_{\mathfrak{F}}([0, T], X).$$

By Lemma 2, the equation

$$\begin{aligned} y(t) &= S(T-t)\xi + \int_t^T S(s-t)F(s, \bar{y}(s), \bar{z}(s))ds \\ &+ \int_t^T S(s-t)[g(s) + z(s)]dw(s), \quad P\text{-a.s.} \end{aligned} \quad (11)$$

has a unique solution in $M_\beta[0, T]$.

Thus, the operator $\Phi : M_\beta[0, T] \rightarrow M_\beta[0, T]$ defined by

$$\Phi(\bar{y}, \bar{z}) = (y, z),$$

where (y, z) is the solution of (11), is well-defined. Moreover, the inequality (4) implies that

$$\begin{aligned} \|\Phi(\bar{y}, \bar{z}) - \Phi(\tilde{y}, \tilde{z})\|_0^2 &\leq 12M_S^2 \frac{1}{\beta} \int_0^T e^{2\beta s} \mathbf{E} \|F(s, \bar{y}(s), \bar{z}(s)) - F(s, \tilde{y}(s), \tilde{z}(s))\|^2 ds \\ &\leq 12M_S^2 L \frac{1}{\beta} \int_0^T e^{2\beta s} \mathbf{E} (\|\bar{y}(s) - \tilde{y}(s)\|^2 + \|\bar{z}(s) - \tilde{z}(s)\|^2) ds \\ &= 12M_S^2 L \frac{1}{\beta} T \|\bar{y}, \bar{z}) - (\tilde{y}, \tilde{z})\|_0^2. \end{aligned}$$

We can choose $\beta > 0$ large enough to get the contractivity of the operator Φ on $M_\beta[0, T]$, which in turn implies the existence and uniqueness of the solution to (2). ■

3.2 Approximation

We now construct an approximate sequence using a Picard-type iteration. Let $y_0(t) = 0$, and let $\{y_n, z_n\}$ be a sequence in $L^2_{\frac{2}{3}}([0, T], X) \times L^2_{\frac{2}{3}}([0, T], L^0_2)$ defined recursively by

$$\begin{aligned} y_n(t) &= S(T-t)\xi + \int_t^T S(s-t)F(s, y_{n-1}(s), z_n(s))ds \\ &\quad + \int_t^T S(s-t)[G(s, y_{n-1}(s)) + z_n(s)]dw(s), \end{aligned} \quad (12)$$

on $0 \leq t \leq T$. We remark that by Theorem 4 the equation (12) has a unique solution (y_n, z_n) .

To state our main result, we impose the following assumptions on the functions F and G .

(N1) $F(\cdot, 0, 0) \in L^2([0, T], X)$, $G(\cdot, 0) \in L^2([0, T], L^0_2)$.

(N2) There exists an $l > 0$ such that

$$\begin{aligned} \|F(t, y, z) - F(t, \bar{y}, \bar{z})\|^2 &\leq \rho(\|y - \bar{y}\|^2) + l\|z - \bar{z}\|^2, \\ \|G(t, y) - G(t, \bar{y})\|^2 &\leq \rho(\|y - \bar{y}\|^2) \end{aligned}$$

for all $t \in [0, T]$, $y, \bar{y} \in X$, $z, \bar{z} \in L^0_2$. Here ρ is a concave increasing function from $[0, \infty)$ to $[0, \infty)$ such that $\rho(0) = 0$, $\rho(u) > 0$ for $u > 0$ and

$$\int_{0^+} \frac{du}{\rho(u)} = \infty.$$

Since ρ is concave and $\rho(0) = 0$, there is a pair of positive numbers a and b such that

$$\rho(u) \leq a + bu \quad (13)$$

for all $u \geq 0$. Therefore, under assumptions (N1) and (N2), $F(\cdot, y(\cdot), z(\cdot)) \in L^2_{\frac{2}{3}}([0, T], X)$ and $G(\cdot, y(\cdot)) \in L^2_{\frac{2}{3}}([0, T], L^0_2)$, whenever $y(\cdot) \in L^2_{\frac{2}{3}}([0, T], X)$ and $z(\cdot) \in L^2_{\frac{2}{3}}([0, T], L^0_2)$.

Now we introduce some important constants used throughout the paper.

$$\begin{aligned} C_1 &= 24M_S^2 \left(e^{2\beta T} \mathbf{E} \|\xi\|^2 + \frac{1}{2\beta} \int_0^T e^{2\beta s} (2\|F(s, 0, 0)\|^2 + 2a) ds \right) \\ &\quad + 2 \int_0^T e^{2\beta s} (2\|G(s, 0)\|^2 + 2a) ds, \quad C_1 \geq 4aT, \\ C_2 &= 24M_S^2 \frac{1}{\beta} b + 4b, \\ C_3 &= \left(12M_S^2 \frac{1}{\beta} + 2 \right) e^{2\beta T}, \quad C_4 = C_3 \rho(4C_1 \exp(C_2 T)). \end{aligned} \quad (14)$$

Lemma 5 Under hypotheses (N1) and (N2), for all $0 \leq t \leq T$ and $n \geq 1$,

$$\mathbf{E} \left(\sup_{t \leq s \leq T} e^{2\beta s} \|y_n(s)\|^2 \right) \leq C_1 \exp(C_2(T-t)), \quad (15)$$

$$\mathbf{E} \int_t^T e^{2\beta s} \|z_n(s)\|^2 ds \leq 2C_1(1 + C_2(T-t) \exp(C_2(T-t))), \quad (16)$$

where C_1 and C_2 are both positive constants defined in (14).

Proof. It follows from Lemma 2 that

$$\begin{aligned} & \mathbf{E} \sup_{t \leq s \leq T} e^{2\beta s} \|y_n(s)\|^2 + \mathbf{E} \int_t^T e^{2\beta s} \|z_n(s)\|^2 ds \\ & \leq 24M_S^2 \left(e^{2\beta T} \mathbf{E} \|\xi\|^2 + \frac{1}{2\beta} \int_t^T e^{2\beta r} \mathbf{E} \|F(r, y_{n-1}(r), z_n(r))\|^2 dr \right) \\ & + 2\mathbf{E} \int_t^T e^{2\beta r} \|G(r, y_{n-1}(r))\|^2 dr. \end{aligned} \quad (17)$$

Using hypotheses (N1) and (N2) with (13) yields

$$\begin{aligned} \|F(s, y_{n-1}(s), z_n(s))\|^2 & \leq 2 \|F(s, 0, 0)\|^2 + 2a + 2b \|y_{n-1}(s)\|^2 + 2l \|z_n(s)\|^2, \\ \|G(s, y_{n-1}(s))\|^2 & \leq 2 \|G(s, 0)\|^2 + 2a + 2b \|y_{n-1}(s)\|^2. \end{aligned}$$

Substituting these into (17) gives

$$\begin{aligned} & \mathbf{E} \sup_{t \leq s \leq T} e^{2\beta s} \|y_n(s)\|^2 + \mathbf{E} \int_t^T e^{2\beta s} \|z_n(s)\|^2 ds \\ & \leq 24M_S^2 e^{2\beta T} \mathbf{E} \|\xi\|^2 \\ & + 24M_S^2 \frac{1}{2\beta} \mathbf{E} \int_t^T e^{2\beta s} \left(2 \|F(s, 0, 0)\|^2 + 2a + 2b \|y_{n-1}(s)\|^2 + 2l \|z_n(s)\|^2 \right) ds \\ & + 2\mathbf{E} \int_t^T e^{2\beta s} \left(2 \|G(s, 0)\|^2 + 2a + 2b \|y_{n-1}(s)\|^2 \right) ds. \end{aligned}$$

Thus

$$\begin{aligned} & \mathbf{E} \sup_{t \leq s \leq T} e^{2\beta s} \|y_n(s)\|^2 + \left(1 - 24M_S^2 \frac{1}{\beta} l \right) \mathbf{E} \int_t^T e^{2\beta s} \|z_n(s)\|^2 ds \\ & \leq C_1 + C_2 \mathbf{E} \int_t^T \sup_{s \leq r \leq T} \left(e^{2\beta r} \|y_{n-1}(r)\|^2 \right) ds, \end{aligned} \quad (18)$$

where C_1 and C_2 are defined in (14). Choosing $\beta > 0$ such that $1 - 24M_S^2 \frac{1}{\beta} l = \frac{1}{2}$, we obtain

$$\sup_{1 \leq n \leq m} \mathbf{E} \left(\sup_{t \leq s \leq T} e^{2\beta s} \|y_n(s)\|^2 \right) \leq C_1 + C_2 \int_t^T \sup_{1 \leq n \leq m} \mathbf{E} \sup_{s \leq r \leq T} \left(e^{2\beta r} \|y_{n-1}(r)\|^2 \right) ds.$$

An application of the Gronwall inequality now implies

$$\sup_{1 \leq n \leq m} \mathbf{E} \left(\sup_{t \leq s \leq T} e^{2\beta s} \|y_n(s)\|^2 \right) \leq C_1 \exp(C_2(T-t)).$$

Since m was arbitrary, the inequality (15) follows. Finally it follows from (18) that for $\beta = 48M_S^2 l$ we have

$$\begin{aligned} \mathbf{E} \int_t^T e^{2\beta s} \|z_n(s)\|^2 ds &\leq 2C_1 + 2C_2 \int_t^T C_1 \exp(C_2(T-s)) ds \\ &\leq 2C_1 (1 + C_2(T-t) \exp(C_2(T-t))). \end{aligned}$$

■

Lemma 6 *Under hypotheses (N1) and (N2), there exists a constant $C_3 > 0$ defined in (14) such that*

$$\mathbf{E} \sup_{t \leq s \leq T} e^{2\beta s} \|y_{n+m}(s) - y_n(s)\|^2 \leq C_3 \int_t^T \rho \left(\mathbf{E} \sup_{s \leq r \leq T} e^{2\beta r} \|y_{n+m-1}(r) - y_{n-1}(r)\|^2 \right) ds$$

for all $0 \leq t \leq T$ and $n, m \geq 1$.

Proof. Applying Lemma 2 we have

$$\begin{aligned} &\mathbf{E} \sup_{t \leq s \leq T} e^{2\beta s} \|y_{n+m}(s) - y_n(s)\|^2 + \mathbf{E} \int_t^T e^{2\beta s} \|z_{n+m}(s) - z_n(s)\|^2 ds \\ &\leq 12M_S^2 \frac{1}{\beta} \mathbf{E} \int_t^T e^{2\beta s} \|F(s, y_{n+m-1}(s), z_{n+m}(s)) - F(s, y_{n-1}(s), z_n(s))\|^2 ds \\ &\quad + 2\mathbf{E} \int_t^T e^{2\beta s} \|G(s, y_{n+m-1}(s)) - G(s, y_{n-1}(s))\|^2 ds \\ &\leq \left(12M_S^2 \frac{1}{\beta} + 2 \right) e^{2\beta T} \mathbf{E} \int_t^T \rho \left(e^{2\beta s} \|y_{n+m-1}(s) - y_{n-1}(s)\|^2 \right) \\ &\quad + 12M_S^2 \frac{1}{\beta} \mathbf{E} \int_t^T e^{2\beta s} \|z_{n+m}(s) - z_n(s)\|^2 ds. \end{aligned}$$

For sufficiently large $\beta > 0$ we have

$$\begin{aligned} &\mathbf{E} \sup_{t \leq s \leq T} e^{2\beta s} \|y_{n+m}(s) - y_n(s)\|^2 + \left(1 - 12M_S^2 \frac{1}{\beta} \right) \mathbf{E} \int_t^T e^{2\beta s} \|z_{n+m}(s) - z_n(s)\|^2 ds \\ &\leq C_3 \int_t^T \rho \left(\mathbf{E} \sup_{s \leq r \leq T} e^{2\beta r} \|y_{n+m-1}(r) - y_{n-1}(r)\|^2 \right) ds. \end{aligned} \quad (19)$$

■

Lemma 7 *Under hypotheses (NI) and (N2), there exists a constant $C_4 > 0$*

defined in (14) such that

$$\mathbf{E} \sup_{t \leq s \leq T} e^{2\beta s} \|y_{n+m}(s) - y_n(s)\|^2 \leq C_4 (T - t).$$

for all $0 \leq t \leq T$ and for all $n, m \geq 1$.

Proof. By Lemmas 5 and 6 we have

$$\begin{aligned} \mathbf{E} \sup_{t \leq s \leq T} e^{2\beta s} \|y_{n+m}(s) - y_n(s)\|^2 &\leq C_3 \int_t^T \rho \left(\mathbf{E} \sup_{s \leq r \leq T} e^{2\beta r} \|y_{n+m-1}(r) - y_{n-1}(r)\|^2 \right) ds \\ &\leq C_3 \int_t^T \rho(4C_1 \exp(C_2(T-s))) ds \\ &\leq C_3 \rho(4C_1 \exp(C_2 T)) (T-t) = C_4 (T-t). \end{aligned}$$

The proof is complete. ■

Define

$$\begin{aligned} \varphi_1(t) &= C_4 (T-t), \\ \varphi_{n+1}(t) &= C_3 \int_t^T \rho(\varphi_n(s)) ds, \quad n \geq 1, \\ \tilde{\varphi}_{n,m}(t) &= \mathbf{E} \sup_{t \leq s \leq T} e^{2\beta s} \|y_{n+m}(s) - y_n(s)\|^2, \quad n \geq 1, m \geq 1. \end{aligned}$$

Lemma 8 *There exists $0 \leq T_1 < T$ such that for all $n, m \geq 1$*

$$0 \leq \tilde{\varphi}_{n,m}(t) \leq \varphi_n(t) \leq \varphi_{n-1}(t) \leq \cdots \leq \varphi_1(t) \quad \text{for all } t \in [T_1, T]. \quad (20)$$

Proof. We prove this lemma by induction in n .

By Lemma 7, we have

$$\tilde{\varphi}_{1,m}(t) = \mathbf{E} \sup_{t \leq s \leq T} e^{2\beta s} \|y_{1+m}(s) - y_1(s)\|^2 \leq C_4 (T-t) = \varphi_1(t).$$

By Lemma 6

$$\begin{aligned} \tilde{\varphi}_{2,m}(t) &= \mathbf{E} \sup_{t \leq s \leq T} e^{2\beta s} \|y_{2+m}(s) - y_2(s)\|^2 \\ &\leq C_3 \int_t^T \rho \left(\mathbf{E} \sup_{s \leq r \leq T} e^{2\beta r} \|y_{1+m}(r) - y_1(r)\|^2 \right) ds \\ &= C_3 \int_t^T \rho(\tilde{\varphi}_{1,m}(s)) ds \leq C_3 \int_t^T \rho(\varphi_1(s)) ds = \varphi_2(t). \end{aligned}$$

We must show that there exists $T_1 > 0$ such that for all $t \in [T_1, T]$ the inequality

$$\varphi_2(t) = C_3 \int_t^T \rho(C_4(T-s)) ds \leq C_4 (T-t) = \varphi_1(t) \quad (21)$$

holds.

To this end, note that this inequality holds provided that

$$C_3 \rho(C_4(T-t)) \leq C_4 = C_3 \rho(4C_1 \exp(C_2 T))$$

or

$$C_3 \rho(4C_1 \exp(C_2 T)) = C_4(T-t) \leq 4C_1 \exp(C_2 T) = 4A.$$

On the other hand, this holds if

$$C_3 \{A + 4bA\}(T-t) \leq 4A.$$

Since $A = C_1 \exp(C_2 T) \geq C_1 \geq 4aT$ the above inequality holds if

$$T-t \leq \frac{4}{C_3 \left[\frac{1}{4T} + 4b \right]} \leq \frac{4}{C_3 \left[\frac{a}{A} + 4b \right]}.$$

Thus, (21) holds for any t satisfying

$$T-t \leq \frac{4}{C_3 \left[\frac{1}{4T} + 4b \right]}.$$

Clearly, such t does not depend on the final value ξ . Thus, there exists $T_1 > 0$ such that

$$\varphi_2(t) \leq \varphi_1(t)$$

for all $t \in [T_1, T]$. Now, assume that (20) holds for some $n \geq 2$. Then, using the same inequalities as above yields

$$\begin{aligned} \tilde{\varphi}_{n+1,m}(t) &\leq C_3 \int_t^T \rho \left(\mathbf{E} \sup_{s \leq r \leq T} e^{2\beta r} \|y_{n+m}(r) - y_n(r)\|^2 \right) ds \\ &= C_3 \int_t^T \rho(\tilde{\varphi}_{n,m}(s)) ds \leq C_3 \int_t^T \rho(\varphi_n(s)) ds = \varphi_{n+1}(t) \end{aligned}$$

for all $t \in [T_1, T]$. On the other hand, we have

$$\varphi_{n+1}(t) = C_3 \int_t^T \rho(\varphi_n(s)) ds \leq C_3 \int_t^T \rho(\varphi_{n-1}(s)) ds = \varphi_n(t)$$

for all $t \in [T_1, T]$. This completes the proof. ■

Theorem 9 *Assume that (N1) and (N2) hold. Then, there exists a unique mild solution (y, z) to the equation (1).*

Proof. *Uniqueness:* To show the uniqueness, let both (y, z) and (\tilde{y}, \tilde{z}) be

solutions of the equation (1). Then, Lemma 2 implies

$$\begin{aligned}
& \mathbf{E} \sup_{t \leq s \leq T} e^{2\beta s} \|y(s) - \tilde{y}(s)\|^2 + \mathbf{E} \int_t^T e^{2\beta s} \|z(s) - \tilde{z}(s)\|^2 ds \\
& \leq 12M_S^2 \frac{1}{\beta} \int_t^T e^{2\beta s} \|F(s, y(s), z(s)) - F(s, \tilde{y}(s), \tilde{z}(s))\|^2 ds \\
& + 2\mathbf{E} \int_t^T e^{2\beta s} \|G(s, y(s)) - G(s, \tilde{y}(s))\|^2 ds \\
& \leq C \int_t^T \rho \left(\mathbf{E} \sup_{s \leq r \leq T} e^{2\beta r} \|y(r) - \tilde{y}(r)\|^2 \right) ds + 12M_S^2 \frac{1}{\beta} \mathbf{E} \int_t^T e^{2\beta s} \|z(s) - \tilde{z}(s)\|^2 ds.
\end{aligned}$$

Therefore, one can apply the Bihari inequality to (4) to obtain

$$\mathbf{E} \sup_{t \leq s \leq T} e^{2\beta s} \|y(s) - \tilde{y}(s)\|^2 = 0.$$

So, $y(t) = \tilde{y}(t)$ for all $0 \leq t \leq T$ almost surely. It then follows from (4) that $z(t) = \tilde{z}(t)$ for all $0 \leq t \leq T$ almost surely as well. This establishes the uniqueness.

Existence: We claim that

$$\mathbf{E} \sup_{t \leq s \leq T} e^{2\beta s} \|y_{n+m}(s) - y_n(s)\|^2 \rightarrow 0, \text{ for all } T_1 \leq t \leq T, \quad (22)$$

as $n, m \rightarrow \infty$. Note that, by definition φ_n is continuous on $[T_1, T]$. Note also that for each $n \geq 1$, $\varphi_n(\cdot)$ is decreasing on $[T_1, T]$, and for each t , $\varphi_n(t)$ is a nonincreasing sequence. Therefore, we can define the function $\varphi(t)$ by $\varphi_n(t) \downarrow \varphi(t)$. It is easy to verify that $\varphi(t)$ is continuous and nonincreasing on $[T_1, T]$. By the definitions of $\varphi_n(t)$ and $\varphi(t)$ we get

$$\varphi(t) = \lim_{n \rightarrow \infty} C_3 \int_t^T \rho(\varphi_n(s)) ds = C_3 \int_t^T \rho(\varphi(s)) ds$$

for each $t \in [T_1, T]$. Since

$$\int_{0^+} \frac{du}{\rho(u)} = \infty$$

the Bihari inequality implies $\varphi(t) = 0$ for all $t \in [T_1, T]$. Consequently, $\lim_{n \rightarrow \infty} \varphi_n(T_1) = 0$. By Lemma 8

$$\begin{aligned}
\mathbf{E} \sup_{t \leq s \leq T} e^{2\beta s} \|y_{n+m}(s) - y_n(s)\|^2 & \leq \sup_{T_1 \leq t \leq T} \tilde{\varphi}_{n,m}(t) \\
& \leq \sup_{T_1 \leq t \leq T} \varphi_n(t) = \varphi_n(T_1) \xrightarrow{n \rightarrow \infty} 0.
\end{aligned}$$

So, (22) must hold. Applying (22) to (19) we see that $\{y_n, z_n\}$ is a Cauchy (hence convergent) sequence in $M_\beta[T_1, T]$; denote the limit by (y, z) . Now

letting $n \rightarrow \infty$ in (12) we obtain

$$y(t) = S(T-t)\xi + \int_t^T S(s-t)F(s, y(s), z(s))ds \\ + \int_t^T S(s-t)[G(s, y(s)) + z(s)]dw(s),$$

on $[T_1, T]$. Since the value of T_1 depends only on the function ρ , one can deduce by iteration the existence on $[T - k(T - T_1), T]$ for each k , and therefore the existence on the entire interval $[0, T]$.

The theorem has been proved. ■

As an illustration of the applicability of this general existence and uniqueness result, we consider examples of concrete backward stochastic partial differential equations.

Example A Let \mathcal{D} be a bounded domain in \mathbb{R}^N with smooth boundary $\partial\mathcal{D}$. Consider the following initial boundary value problem.

$$\begin{aligned} \partial y(t, x) &= (\Delta_z z(t, x) + F(t, x, y(t, x), z(t, x))) \partial t \\ &+ [G(t, x, y(t, x)) + z(t, x)] d\beta(t), \text{ a.e. on } (0, T) \times \mathcal{D} \\ y(t, x) &= 0, \text{ a.e. on } (0, T) \times \partial\mathcal{D}, \\ y(T, x) &= \xi(T, x), \text{ a.e. on } \mathcal{D}, \end{aligned} \tag{23}$$

where $y : [0, T] \times \mathcal{D} \rightarrow \mathbb{R}$, $z : [0, T] \times \mathcal{D} \rightarrow L_2^0(\mathbb{R}^N; L^2(\mathcal{D}))$, $F : [0, T] \times \mathcal{D} \times \mathbb{R} \times L_2^0(\mathbb{R}^N; L^2(\mathcal{D})) \rightarrow \mathbb{R}$, $G : [0, T] \times \mathcal{D} \times \mathbb{R} \rightarrow L_2^0(\mathbb{R}^N, L^2(\mathcal{D}))$, β is a standard N -dimensional Brownian motion (equipped with a normal filtration $\{\mathfrak{F}_t\}$, and $\xi : [0, T] \times \mathcal{D} \rightarrow \mathbb{R}$ is an \mathfrak{F}_T -measurable random variable.

We impose the following conditions:

(E1) F satisfies the Caratheodory conditions (i.e., measurable in (t, x, y) and continuous in the fourth variable) and there exists $M_F > 0$ such that

$$|F(t, x, w_1, y_1) - F_1(t, x, w_2, y_2)| \leq M_F [|w_1 - w_2| + \|y_1 - y_2\|_{L_2^0(\mathbb{R}^N, L^2(\mathcal{D}))}],$$

for all $0 \leq t \leq T, x \in \mathcal{D}, w_1, w_2 \in \mathbb{R}, y_1, y_2 \in L_2^0(\mathbb{R}^N, L^2(\mathcal{D}))$.

(E2) G satisfies the Caratheodory conditions and there exists $M_G > 0$ such that

$$\|G(t, x, w_1) - G(t, x, w_2)\|_{L_2^0(\mathbb{R}^N, L^2(\mathcal{D}))} \leq M_G |w_1 - w_2|,$$

for all $0 \leq t \leq T, z \in \mathcal{D}, w_1, w_2 \in \mathbb{R}$.

We have the following theorem:

Theorem 10 *If (E1) and (E2) are satisfied, then (23) has a unique mild solution $(y, z) \in L^2(0, T; L^2(\Omega, L^2(\mathcal{D}))) \times L_F^2(0, T; L^2(\mathbb{R}^N, L^2(\Omega, L^2(\mathcal{D}))))$.*

Proof. Let $X = L^2(\mathcal{D})$ and $K = \mathbb{R}^N$. Also, denote $\frac{\partial y}{\partial t}$ by $y'(t)$, and define the operator A by

$$Ay(t, \cdot) = \Delta_x y(t, \cdot), \quad y \in H^2(\mathcal{D}) \cap H_0^1(\mathcal{D}). \quad (24)$$

It is known that A generates a strongly continuous semigroup $\{S(t)\}$ on $L^2(\mathcal{D})$ (see [15]). Define the maps $f : [0, T] \times X \times L_2^0(K; X) \rightarrow X$ and $g : [0, T] \times X \rightarrow L_2^0(K, X)$ by

$$f(t, y(t), z(t))(x) = F(t, x, y(t, x), z(t, x)) \quad (25)$$

$$g(t, y(t))(x) = G(t, x, y(t, x)), \quad (26)$$

for all $0 \leq t \leq T, x \in \mathcal{D}$. With these identifications, we observe that (23) can be written in the abstract form (1). Clearly, f and g as defined in (25) and (26) satisfy (N1) and (N2), respectively. Hence, we can invoke Theorem 9 to conclude that (23) has a unique mild solution $(y, z) \in L^2(0, T; L^2(\Omega, L^2(\mathcal{D}))) \times L_F^2(0, T; L^2(\mathbb{R}^N, L^2(\Omega, L^2(\mathcal{D}))))$. ■

4 Stochastic Maximum Principle

In this section we consider the following stochastic controlled system

$$\begin{aligned} y(t) + \int_t^T S(s-t) f(s, y(s), z(s), u(s)) ds \\ + \int_t^T S(s-t) z(s) dw(s) = S(T-t) \xi. \end{aligned} \quad (27)$$

with the cost functional

$$J(u) = \mathbf{E}h(y(0)) + \mathbf{E} \int_0^T l(t, y(t), z(t), u(t)) dt. \quad (28)$$

Here, $f : [0, T] \times X \times L_2^0 \times U \rightarrow X$, $h : X \rightarrow X$, $l : [0, T] \times X \times L_2^0 \times U \rightarrow \mathbb{R}$ are measurable functions, $\xi \in L^2(\Omega, \mathfrak{F}_T, X)$ and $u : [0, T] \times \Omega \rightarrow U$.

We impose the following assumptions.

- (A1) f, l, h are continuously differentiable with respect to (y, z) .
- (A2) The derivatives of f with respect to y, z are uniformly bounded

$$\|f_y\| + \|f_z\| \leq C$$

and

$$\|h_y\| + \|l_y\| \leq C(1 + \|y\|), \quad \|l_z\| \leq C(1 + \|z\|),$$

where C is a positive constant.

Now we define

$$U_{ad} = \left\{ u \in L^2_{\mathfrak{F}}([0, T], U) : u(t, \omega) \in U \right\}.$$

It is clear that under (A1)-(A2) for any $u \in U_{ad}$ the state equation (27) admits a unique solution $(y, z) = (y(\cdot, u(\cdot)), z(\cdot, u(\cdot)))$ and the cost functional (28) is well-defined. We call (y, z, u) an admissible triple, and (y, z) an admissible state process. An optimal control problem can be stated as follows.

Problem A. Find a control $u^0(\cdot) \in U_{ad}$ such that

$$J(u^0) = \inf_{u \in U_{ad}} J(u). \quad (29)$$

Any control u^0 satisfying the equality (29) is called an optimal control. The corresponding $(y^0, z^0) = (y(\cdot, u^0(\cdot)), z(\cdot, u^0(\cdot)))$ and (y^0, z^0, u^0) are called an optimal state process and optimal triple, respectively.

Assume that

$$(y^0(\cdot), z^0(\cdot), u^0(\cdot))$$

is an optimal solution of the control problem (28) and (27). Consider the following forward stochastic equation

$$\begin{aligned} \psi(t) &= S(t) h_y(y^0(0)) + \int_0^t S(s-t) \{ f_y^*[s] \psi(s) + l_y[s] \} ds \\ &\quad + \int_0^t S(s-t) \{ f_z^*[s] \psi(s) + l_z[s] \} dw(s). \end{aligned} \quad (30)$$

In what follows we shall use the following notations.

$$\begin{aligned} F[t] &= F(t, y^0(t), z^0(t), u^0(t)), \\ \Delta_u F(t) &= F(t, y^0(t), z^0(t), u(t)) - F[t], \\ \Delta_y F(t) &= F(t, y(t), z^0(t), u^0(t)) - F[t], \\ \Delta_z F(t) &= F(t, y^0(t), z(t), u^0(t)) - F[t]. \end{aligned}$$

Let H be a Hamiltonian function given by

$$H(t, v) = \langle f(t, y^0(t), z^0(t), v), \psi(t) \rangle - l(t, y^0(t), z^0(t), v).$$

For any given

$$v \in U_{ad}, t_0 \in [0, T], 0 < \varepsilon \leq T - t_0,$$

define a spike variational control by

$$u^\varepsilon(t) = \begin{cases} v, & t \in [t_0, t_0 + \varepsilon], \\ u^0(t), & \text{otherwise.} \end{cases}$$

Let $(y^\varepsilon(\cdot), z^\varepsilon(\cdot))$ be the solution of (27) corresponding to the admissible control $u^\varepsilon(\cdot)$, and let $(p^\varepsilon(\cdot), q^\varepsilon(\cdot))$ be the solution of the following linear BSDE

$$\begin{aligned} p^\varepsilon(t) + \int_t^T S(s-t) f_y[s] p^\varepsilon(s) ds + \int_t^T S(s-t) f_z[s] q^\varepsilon(s) ds \\ + \int_t^T S(s-t) q^\varepsilon(s) dw(s) + \int_t^T S(s-t) \Delta_{u^\varepsilon} f[s] ds = 0. \end{aligned} \quad (31)$$

We have the following theorem.

Theorem 11 *Let (A1) and (A2) hold. Then*

$$\sup_{0 \leq t \leq T} \mathbf{E} \|p^\varepsilon(t)\|^2 + \mathbf{E} \int_0^T \|q^\varepsilon(t)\|^2 dt = O(\varepsilon^2), \quad (32)$$

$$\sup_{0 \leq t \leq T} \mathbf{E} \|p^\varepsilon(t)\|^4 + \mathbf{E} \int_0^T \|q^\varepsilon(t)\|^4 dt = O(\varepsilon^4), \quad (33)$$

$$\sup_{0 \leq t \leq T} \mathbf{E} \|y^\varepsilon(t) - y^0(t) - p^\varepsilon(t)\|^2 + \mathbf{E} \int_0^T \|z^\varepsilon(t) - z^0(t) - q^\varepsilon(t)\|^2 dt = o(\varepsilon^2). \quad (34)$$

Moreover, the following formula holds

$$\begin{aligned} J(u^\varepsilon) - J(u^0) \\ = \mathbf{E} \langle h_y(y^0(0)), p^\varepsilon(0) \rangle + \mathbf{E} \int_0^T \langle l_y[s], p^\varepsilon(s) \rangle ds \\ + \mathbf{E} \int_0^T \langle l_z[s], q^\varepsilon(s) \rangle ds + \mathbf{E} \int_0^T \Delta l_u[s] ds + o(\varepsilon). \end{aligned} \quad (35)$$

Proof. By the Taylor formula we have

$$\begin{aligned}
& J(u^\varepsilon) - J(u^0) \\
&= \mathbf{E} \left[h(y^\varepsilon(0)) - h(y^0(0)) \right] \\
&+ \mathbf{E} \int_0^T \left[l(s, y^\varepsilon(s), z^\varepsilon(s), u^\varepsilon(s)) - l(s, y^0(s), z^0(s), u^0(s)) \right] ds \\
&= \mathbf{E} \int_0^1 \left\langle h_y \left(y^0(0) + \theta \left(y^\varepsilon(0) - y^0(0) \right) \right), \Delta y^0(0) \right\rangle d\theta \\
&+ \mathbf{E} \int_0^T \left\langle l_y \left(s, y^0(s) + \theta \left(y^\varepsilon(s) - y^0(s) \right), z^\varepsilon(s), u^\varepsilon(s) \right), \Delta y^0(s) \right\rangle ds \\
&+ \mathbf{E} \int_0^T \left\langle l_z \left(s, y^0(s), z^0(s) + \theta \left(z^\varepsilon(s) - z^0(s) \right), u^\varepsilon(s) \right), \Delta z^0(s) \right\rangle ds,
\end{aligned}$$

where $\Delta y^0(s) = y^\varepsilon(s) - y^0(s)$ and $\Delta z^0(s) = z^\varepsilon(s) - z^0(s)$. Let $Y^\varepsilon(t) = y^\varepsilon(t) - y^0(t) - p^\varepsilon(t)$ and $Z^\varepsilon(t) = z^\varepsilon(t) - z^0(t) - q^\varepsilon(t)$. Then

$$\begin{aligned}
& J(u^\varepsilon) - J(u^0) \\
&= \mathbf{E} \left\langle h_y \left(y^0(0) \right), p^\varepsilon(0) \right\rangle + \mathbf{E} \left\langle h_y \left(y^0(0) \right), Y^\varepsilon(0) \right\rangle \\
&+ \mathbf{E} \int_0^1 \left\langle h_y \left(y^0(0) + \theta \left(y^\varepsilon(0) - y^0(0) \right) \right) - h_y \left(y^0(0) \right), p^\varepsilon(0) + Y^\varepsilon(0) \right\rangle d\theta \\
&+ \mathbf{E} \int_0^T \left\langle l_y [s], p^\varepsilon(s) + Y^\varepsilon(s) \right\rangle ds \\
&+ \mathbf{E} \int_0^T \left\langle l_z [s], q^\varepsilon(s) + Z^\varepsilon(s) \right\rangle ds + \mathbf{E} \int_0^T \Delta l_u [s] ds \\
&+ \mathbf{E} \int_0^T \left\langle l_y \left(s, y^0(s) + \theta \left(y^\varepsilon(s) - y^0(s) \right), z^\varepsilon(s), u^\varepsilon(s) \right) - l_y [s], p^\varepsilon(s) + Y^\varepsilon(s) \right\rangle ds \\
&+ \mathbf{E} \int_0^T \left\langle l_z \left(s, y^0(s), z^0(s) + \theta \left(z^\varepsilon(s) - z^0(s) \right), u^\varepsilon(s) \right) - l_z [s], q^\varepsilon(s) + Z^\varepsilon(s) \right\rangle ds.
\end{aligned}$$

Then by Theorem 11, we can obtain (35). ■

Theorem 12 (*Stochastic Maximum Principle*) Assume that (A1) and (A2) hold, and let (y^0, z^0, u^0) be an optimal triple of Problem A. Then, there is a process ψ satisfying (30) such that

$$H(t, v) \leq H(t, u^0(t)), \quad (36)$$

for all $v \in U$, a.e. $t \in [0, T]$, P -a.s.

Proof. By the formula (35) we have

$$\begin{aligned}
& J(u^\varepsilon) - J(u^0) \\
&= \mathbf{E} \left\langle h_y \left(y^0(0) \right), p^\varepsilon(0) \right\rangle + \mathbf{E} \int_0^T \left\langle l_y [s], p^\varepsilon(s) \right\rangle ds \\
&+ \mathbf{E} \int_0^T \left\langle l_z [s], q^\varepsilon(s) \right\rangle ds + \mathbf{E} \int_0^T \Delta l_u [s] ds + o(\varepsilon).
\end{aligned}$$

On the other hand

$$\begin{aligned} & \mathbf{E} \langle h_y(y^0(0)), p^\varepsilon(0) \rangle + \mathbf{E} \int_0^T \langle l_y[s], p^\varepsilon(s) \rangle ds + \mathbf{E} \int_0^T \langle l_z[s], q^\varepsilon(s) \rangle ds \\ & = \mathbf{E} \int_0^T \langle \Delta_{u^\varepsilon} f[s], p^\varepsilon(s) \rangle ds. \end{aligned}$$

Thus

$$0 \leq J(u^\varepsilon) - J(u^0) = \mathbf{E} \int_0^T \langle \Delta_{u^\varepsilon} f[s], p^\varepsilon(s) \rangle ds + \mathbf{E} \int_0^T \Delta l_u[s] ds + o(\varepsilon)$$

and from here we can easily obtain the variational inequality (36). ■

5 A backward linear quadratic problem

In this section we apply Theorem 12 to a linear quadratic problem as a particular case of the Problem A.

Consider the following problem

$$J(u) = \mathbf{E} \langle Gy(0), y(0) \rangle + \mathbf{E} \int_0^T \langle \Gamma(t) u(t), u(t) \rangle dt \rightarrow \min \quad (37)$$

subject to

$$\begin{cases} dy(t) = [Ay(t) + Bu(t) + Cz(t)] dt \\ \quad \quad \quad + z(t) dw(t), \\ y(T) = \xi. \end{cases} \quad (38)$$

Here $B : U \rightarrow X$, $C : L_2^0 \rightarrow X$, $\Gamma : [0, T] \rightarrow L(U)$. We assume that $G = G^*$, $\Gamma(t) = \Gamma^*(t) \geq \gamma I$.

Let u^0 be an optimal control, and (y^0, z^0) be the corresponding state process. The adjoint process ψ is the solution of

$$\begin{aligned} -d\psi(t) &= A^* \psi(t) dt + C^* \psi(t) dw(t), \\ \psi(0) &= Gz^0(0). \end{aligned} \quad (39)$$

Theorem 13 *There exists a unique optimal control u^0 for the problem (37)-(38) in the class U_{ad} . Moreover, u^0 has the following representation*

$$u^0(t) = \Gamma^{-1}(t) B^* \psi(t). \quad (40)$$

Proof. It is clear that (37) is a positive quadratic functional of control because of the assumptions on G and $\Gamma(t)$. Hence an optimal control exists.

Furthermore, the BSDE has the form (39). By Theorem 12

$$\langle Bu^0(t), \psi(t) \rangle - \langle \Gamma(t) u^0(t), u^0(t) \rangle \geq \langle Bv, \psi(t) \rangle - \langle \Gamma(t) v, v \rangle$$

for all $v \in U$. This in turn implies (40). Hence the control (40) is the only control which satisfies the stochastic maximum principle. It then must be the optimal control. This completes the proof. ■

Theorem 14 *The optimal control u^0 for the problem (37)-(38) can be represented as*

$$u^0(t) = \Gamma^{-1}(t) B^* P(t) y(t), \quad (41)$$

where $y(t)$ is the solution of

$$\begin{aligned} y'(t) &= [A + B\Gamma^{-1}(t) B^* P(t)] y(t), \\ y(0) &= y^0(0), \end{aligned} \quad (42)$$

and $P(t)$ is the mild solution of

$$\begin{aligned} dP(t) &= - \left(P(t) A + A^* P(t) + P(t) B\Gamma^{-1} B^* P(t) \right) dt \\ &\quad + C^* P(t) dw(t). \end{aligned}$$

Proof. Let $\tilde{\psi}(t) = P(t) y(t)$. We have

$$\begin{aligned} d\tilde{\psi}(t) &= dP(t) y(t) + P(t) dy(t) \\ &= - \left(P(t) A + A^* P(t) + P(t) B\Gamma^{-1}(t) B^* P(t) \right) y(t) dt \\ &\quad - C^* P(t) dw(t) y(t) \\ &\quad + P(t) \left(A + B\Gamma^{-1}(t) B^* \right) y(t) dt \\ &= - \left(A^* P(t) - C^* P(t) dw(t) \right) y(t) \\ &= -A^* \tilde{\psi}(t) dt - C^* \tilde{\psi}(t) dw(t) \end{aligned}$$

So $\tilde{\psi}(t)$ satisfies the same equation as $\psi(t)$. Hence $\tilde{\psi}(t) = \psi(t)$ by the uniqueness. The theorem is proved. ■

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Response to the reviewer's comments on NA-D-05-00322

First, we would like to sincerely thank the reviewer for carefully reading the manuscript and providing valuable comments that have led to improved results. In addition, we would like to address each of the reviewer's comments individually to explain how we modified the paper accordingly.

Comment 1. *In view of the interdisciplinary nature of the journal NA TMA, it would be better if the authors would explain with some details the role of the generator A in Eq(1) and its connection with semigroup $S(t)$.*

Role of the generator in Eq(1) is explained in the example A of the section 3.

Comment 2. *Before the definition 1 it would be better to describe clearly the domain and the range of functions F and G .*

The following sentence is inserted before definition 1: "**where $A : D(A) \subset X \rightarrow X$ is a linear operator which generates a C_0 -semigroup $\{S(t), 0 \leq t \leq T\}$ on X , $F : [0, T] \times X \times L_2^0 \rightarrow X$ and $G : [0, T] \times X \rightarrow L_2^0$ are given mappings, and $\xi \in L^2(\Omega, \mathfrak{F}_T, X)$.**"

Comment 3. *The constant M_S in inequality (7) (p. 4) should be described precisely.*

In order to describe M_S the following sentence is inserted to Lemma 2: "**where $M_S = \sup \{ \|S(t)\|_{\mathfrak{B}(X)}, 0 \leq t \leq T \}$ and $B(X)$ is the space of bounded, linear operators on X .**"

Comment 4. *In connection with remark "...by Theorem 4..." about recursive equation (12) (p.7), I think that it would be better to consider Theorem 4 with equation:*

$$y(t) = S(T-t)\xi + \int_t^T S(s-t)F(s, y(s), z(s))ds + \int_t^T S(s-t)(g(s) + z(s))dw(s)$$

instead of BSEE (2). The proof seems to remain the same.

We have changed Equations (2) and (12) by the equation suggested by reviewer. The reviewer is right the proof remains same.